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Robustness of Three Methods for Assessing Second Order Interaction in Multidimensional Contingency Tables with Small Samples by a Monte Carlo Technique

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ROBUSTNESS OF THREE METHODS FOR ASSESSING SECOND ORDER INTERACTION IN MULTIDIMENSIONAL CONTINGENCY TABLES WITH SMALL SAMPLES BY A MONTE CARLO TECHNIQUE

by

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VITA

The author, Joseph W. Fidler, is the son of Joseph L. Fidler and Jolana (Gajdos) Fidler. He was born June 11, 1943, in Chicago, Illinois.

His elementary education was obtained in the paraochial schools in Chicago. His secondary education was obtained at St. Ignatius High School in Chicago.

In September, 1961, he entered St. Mary's College, Winona, Minnesota and in June, 1965, received the degree of Bachelor of Arts with a major in mathematics. In September, 1967 he entered the graduate school at DePaul University of Chicago and in February, 1970, received the degree of Master of Science in mathematics. In 1979 he held a graduate assistantship at Loyola University of Chicago.

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CHAPTER I

INTRODUCTION

When conducting investigations in education, sociology, or psychology, the researcher attempts to employ variables which represent continuous data as opposed to discrete data. Statistical tools which are based upon the use of continuous measures are more powerful than those based upon the use of discrete data. However, often the experiment does not lend itself to continuous measures and hence the researcher has no choice but to use nonparametric statistics. Typically, this occurs in demographic studies where the only available data are the frequency counts resulting from categorical variables such as political party affiliation or socio-economic status.

Among the nonparametric statistics, the analysis of categorical data by means of contingency tables is prevalent throughout the literature. Given a two-way table, $(R \times C)$, the typical tests for the presence of first order or two factor interaction are Pearson's:

$$X^2=\Sigma(\text{observed} - \text{expected})^2/\text{expected} \qquad (1)$$

and Fisher's:

$$L^2=2\Sigma\text{observed}\{Ln(\text{observed}/\text{expected})\} \qquad (2)$$

(Haberman, 1978). Both statistics are distributed approximately as a χ^2 random variable with the appropriate

degrees of freedom.

With respect to multidimensional contingency tables, the nature of interaction can grow in terms of complexity. In addition to first order interaction, there exist, depending on the number of dimensions, the possibility of various forms of higher order interaction. In particular, the researcher often is interested in detecting the presence of second order interaction among three categorical variables or attributes.

The investigation of second order interaction in three-way contingency tables was initiated in 1935 by Bartlett. Since then a number of methods were derived for testing this type of interaction. For the most part these methods were formulated so as to reduce the tedious calculations that accompanied the Bartlett procedure. Ku and Kullback (1968) gave nine examples of second order interaction in which they compared the results of their method to those of Bartlett, Goodman, Koch, Kastenbaum, Darroch, and Plackett. More recently, Deming and Stephan's iterative proportional fitting technique has been applied to loglinear models of contingency tables (Bishop, Fienberg, & Holland, 1975). The examples cited deal with large samples. Large in this sense is usually taken to mean cell expectations of ten (Lewis, 1950) or five (Edwards, 1950).

This brings us to the purpose of the present study.

How robust are different methods (e.g. Bartlett, Goodman, Iterative Proportional Fitting with Loglinear Models) for testing second order interaction when the assumption of a large sample is violated? Robust refers to how well the statistical procedure performs when some of the underlying assumptions are not completely satisfied. This question is of particular importance to the sociological or educational researcher in that with multidimensional contingency tables, it may be difficult to fulfill the requirement of having a large sample.

Due to practical considerations it is physically impossible to study every method in conjunction with all higher dimensional tables. This study is restricted to the Bartlett method, the Goodman method, and the Iterative Proportional Fitting method as applied to loglinear analysis. Each method was investigated with respect to three dimensions: $2 \times 2 \times 2$, $2 \times 2 \times 3$, and $3 \times 3 \times 3$. Since the question of how small is small is of major importance, it was necessary to consider these tables over various sample sizes. Therefore, the present investigation was designed to include sample sizes of 20, 40, 60, 80, and 100. Finally, it should be pointed out that the skewness of a distribution might be a factor which could influence the results of this investigation. For this reason, the investigation was conducted over various multinomial distributions that theoretically exhibit no second order

interaction. The distributions used are related to the dimensionality of the table under consideration.

The conclusions drawn are empirical in nature. The questions which concern sample size, dimensionality, and skewness of distribution do not lend themselves to analytic methods. Therefore, a Monte Carlo procedure, was employed as a alternative. In a Monte Carlo procedure, a computer is used to randomly generate data. For this study, a computer randomly generated 1,825 contingency tables for each combination of dimension, sample size, and multinomial distribution. Each table was evaluated according to the three methods of testing for second order interaction and a decision to reject or not reject "H_o: no second order interaction" was made at the nominal 5% level. The conclusions drawn from this study were based upon the percent of time H_o was rejected.

In summary, the study attempts to answer empirically the following questions:

(1) How robust are the Bartlett, Goodman, and Iterative Proportional Fitting methods for small sample sizes?

(2) Is robustness affected by the dimensionality of the contingency table and/or the skewness of the underlying multinomial distribution?

The variables that were manipulated in the investigation are:

(1) Method of testing for second order interaction

(2) Sample size of the table

(3) Dimensionality of the table

(4) Skewness of the distribution

Chapter Two contains a review of the literature concerning contingency tables as they are related to methods of testing for second order interaction and the work done on small sample sizes. In Chapter Three there is a detailed description of the design and the rationale for choosing the levels of the variables previously cited. Chapter Four contains the results of the investigation and the interpretation of these results. Finally, in Chapter Five there is a summary of the results, the conclusions that were drawn, and suggestions for further studies. The computer program that was used in the study can be found in Appendix D.

CHAPTER II

REVIEW OF RELATED LITERATURE

Introduction

This chapter reviews the literature relevant to the concepts of second order interaction in a three dimensional contingency table. The review consists of two parts: the definitions and tests of second order interaction in a multi-dimensional contingency table, and the research conducted on contingency tables that have small samples.

Part one deals with definitions of second order interaction and test statistics of second order interaction from the inception of this concept by Bartlett in 1935 to the present. The discussion deals specifically with interpretations (Bartlett, Goodman, Iterative Proportional Fitting) of second order interaction and various statistics (e.g. Goodman's statistic) derived for testing the null hypothesis of no second order interaction in contingency tables.

Part two shows that most studies of contingency tables which used small samples were conducted for the purpose of determining the appropriateness of using the χ^2 distribution as an approximation to the multinomial distribution when the sample is small. For many of the small sample studies, Monte Carlo techniques were used to derive experimentally the distributions that were under consideration. Monte Carlo

procedures were used because the amount of calculation necessary to compute exact distributions was too large to be done feasibly by enumeration.

Concepts of Second Order Interaction

Bartlett Method

The analysis of higher dimensional contingency tables, that is, tables with more than two classifications began in 1935 (Bartlett, 1935). Bartlett said that for a $2 \times 2 \times 2$ contingency table, the analysis of first order interaction is conducted in essentially the same manner as that for a 2×2 contingency table with the third classification ignored. Therefore, the only other question that must be answered is whether or not there is any second order interaction. Before discussing the procedure used by Bartlett to test the null hypothesis of no second order interaction, it would be instructive to bridge the gap between first order interaction and second order interaction in a contingency table.

Suppose a 2×2 contingency table with fixed marginal totals has the classifications A and B. For each cell in the table, let p_{ij} denote the probability of an observation falling in the i^{th} row and j^{th} column. Let p_i represent the probability of an observation falling in the i^{th} row and p_j represent the probability of an observation falling in the j^{th} column. If A and B are independent, then $p_{ij} = p_i p_j$ for all i, j. If A and B are not independent, then there is

first order interaction between A and B (Mayo, 1961). Therefore, the null hypothesis of independence between A and B can be stated $p_{ij} = p_i p_j$. When $p_{ij} \neq p_i p_j$, the more the interaction, then the more the values will differ.

For every combination of i and j, there will be one equation of the form $p_{ij} = p_i p_j$. In a 2 × 2 contingency table, this results in four equations. Through algebraic substitutions, the four equations can be reduced to one equation in terms of the p_{ij}'s. The null hypothesis tested for a 2 × 2 contingency table becomes: $(p_1/p_2)/(p_3/p_4) = 1$ where $p_1=p_{11}$, $p_2=p_{12}$, $p_3=p_{21}$, and $p_4=p_{22}$ (Anderson & Bancroft, 1952). The extent to which $(p_1/p_2)/(p_3/p_4)$ departs from unity represents the degree of association between A and B.

There are a number of techniques available to test the statistical significance of a 2 × 2 contingency table. If the sample is small, the exact probability of each sample outcome can be found through the multinomial distribution. This, in turn leads to a rejection of no first order interaction at some level of significance. If the sample is large, then X^2 (see Equation 1) can be used to test the null hypothesis of no first order interaction. When the null hypothesis is true, X^2 is distributed asymptotically as a χ^2 distribution on (2-1)(2-1) degrees of freedom.

Suppose a 2 × 2 × 2 contingency table with fixed marginal totals has the classifications A, B, and C. As in the 2 × 2 contingency table n_i = observed frequency,

m_i= expected frequency, and p_i= probability of an observation falling into cell i with i=1,2,3,...,8. The table can be represented schematically in the following manner:

	C_1		C_2	
	B_1	B_2	B_1	B_2
A_1	n_1	n_2	n_5	n_6
	p_1	p_2	p_5	p_6
A_2	n_3	n_4	n_7	n_8
	p_3	p_4	p_7	p_8

The amount of first order interaction in C_1 is measured by $(p_1/p_2)/(p_3/p_4)$. Now, if the first order interaction is the same in both levels of C, then there is no second order interaction (Anderson & Bancroft, 1952). This translates into:

$$(p_1/p_2)/(p_3/p_4) = (p_5/p_6)/(p_7/p_8) \quad (3)$$

By simple algebraic manipulatons Equation 3 reduces to:

$$p_1p_4p_6p_7 = p_2p_3p_5p_8 \quad (4)$$

This is precisely the equation given by Bartlett to test the hypothesis of no second order interaction (Bartlett, 1935).

In order to test the statistical significance of Equation 4, the multinomial distribution can be used to calculate the exact probability. However, if the sample is quite large, then the multinomial approach becomes impractical. The alternative suggested by R. A. Fisher to

Bartlett was to solve the cubic equation:

$$(n_1+x)(n_4+x)(n_6+x)(n_7+x)=(n_2-x)(n_3-x)(n_5-x)(n_8-x) \quad (5)$$

where x represents the deviation from expectation in each cell (Bartlett, 1935). The equation can be solved iteratively by some form of numerical methods. The iterative procedure continues until x achieves the desired degree of accuracy.

Once x is determined, the m_i's are found according to the factors in Equation 5. The Pearson statistic (see Equation 1) can be used to test statistical significance on (2-1)(2-1)(2-1) degrees of freedom.

In 1945, Norton reviewed the Bartlett procedure and gave a computational algorithm that extended the Bartlett method for 2 × 2 × 2 contingency tables to contingency tables of the form 2 × 2 × L. Norton said that for each level k, where k=1,2,...,L, of the third order of classification, L quantities x_k must be found such that the first order interactions are equal across the L levels and $\Sigma x_k=0$. x_k is the deviation from expectation in each cell for a given level of the third classification. Each x_k can then be applied to the observed values of the k^{th} level so as to determine the expected values for the k^{th} level. After the expected values have been found, the Pearson statistic (see Equation 1) can be used to test the null hypothesis that there is no second order interaction. Since $\Sigma x_k=0$, only (L-1) number of x_k's must be determined, and hence

there are only (L-1) Bartlett-type cubic equations that must be solved. Norton solved this system by using the same iterative procedure given in the original Bartlett method.

Ten years later, Roy and Kastenbaum (1956) gave mathematical formulas that indicated the mechanism behind the Bartlett method and the Norton method. Roy and Kastenbaum's approach to second order interaction differed from Bartlett's and Norton's procedures in one aspect. Bartlett's and Norton's approaches are based upon analysis of variance situations with fixed marginals along at least two classifications. The Roy and Kastenbaum procedure employed techniques that required only that the table size be fixed.

Roy and Kastenbaum developed their formulas by means of a generalized likelihood function of the cell probabilities in an $R \times C \times L$ contingency table. By maximizing the likelihood function with respect to a set of constraints, they proved the existence of a unique solution to the set of maximum likelihood equations. Roy and Kastenbaum demonstrated that for the $2 \times 2 \times 2$ contingency table, their results were identical to Bartlett's concept of second order interaction. In order to estimate the cell probabilities, they used Lagrange multipliers and derived the same cubic equation that Fisher gave to Bartlett for the $2 \times 2 \times 2$ contingency table.

The generalization from the $2 \times 2 \times 2$ contingency table to the $2 \times 2 \times L$ contingency table and eventually to the $R \times C \times L$ contingency table was an exercise in: listing

the number of constraints on the cell probabilities, forming the appropriate system of equations, solving the system for the set of unique Lagrange multipliers, computing the expected values, and testing the hypothesis of no second order interaction by means of their statistic:

$$X^2_{RK} = \Sigma \mu^2_{ijk} / (n_{ijk} + \eta_{ijk} \mu_{ijk}) \qquad (6)$$

This statistic is distributed asymptotically as a χ^2 distribution on (R-1)(C-1)(L-1) degrees of freedom where: μ_{ijk} is the Lagrange multiplier, n_{ijk} is the observed value, and η_{ijk} is a function of i,j,k which will have a value of ± 1.

The Lagrange multipliers are nothing more than the set of x_ks that Norton used in his procedure. Although there are RCL Lagrange multipliers, by an argument similar to that used by Norton, only (R-1)(C-1)(L-1) of them must be determined. Hence there will be (R-1)(C-1)(L-1) cubic equations in as many unknowns that must be solved. The reduction in the number of unknowns occurs as a result of the number of constraints placed on the cell probabilities.

It was not until 1959 that Kastenbaum and Lamphiear demonstrated a technique which solves the (R-1)(C-1)(L-1) system of equations. The algorithm uses the iterative procedure suggested by Norton which is, for all practical purposes, Newton's method of functional iteration. The procedure is easily adaptable to high speed digital computers and consequently the accuracy of the solution is restricted by only the physical constraints of the computer used. The

criterion used by Roy and Kastenbaum and Kastenbaum and Lamphiear for no second order interaction was:

$$p_{RCL}p_{ijL}/(p_{iCL}p_{RjL}) = p_{RCk}p_{ijk}/(p_{iCk}p_{Rjk}) \qquad (7)$$

for $i=1,2,\ldots,(R-1)$, $j=1,2,\ldots,(C-1)$, and $k=1,2,\ldots,(L-1)$.

So, for the period of 1935 to about 1960, except for Bartlett's work and its extensions by Norton, Roy, Kastenbaum, and Lamphiear, very little research was conducted on multi-dimensional contingency tables. As Mayo said in 1961:

> When it comes to the case of contingency tables with three or more attributes, the textbooks are even barer. There is some periodical literature on higher order interactions, but one must look through many widespread sources to get a meaningful picture. There are several different techniques of testing higher-order interactions and they do not always give comparable results. Very little explanation of the nature and meaning of higher order interactions has been given or widely publicized. (Mayo, 1961, p. 840)

One year later, Lewis echoed Mayo's observations which dealt with the neglect of treatment of higher dimensional contingency tables (Lewis, 1962).

Goodman Method

In 1955 Woolf reviewed the results of a study by Aird, Bentall, and Roberts on the estimation of the relationship between blood group and disease. Woolf pointed out that the difference in proportion of a given blood group in the disease and the control series was not invariant from one community to another. He believed that an estimation of the ratio of one rate to another rate was a better indication of the relationship between blood group and the incidence of a disease.

Woolf was the first to perform a logarithmic transformation on a maximum-likelihood estimate of a ratio (Lindley, 1964). He did this in order to eliminate problems that could arise because of symmetry. Although this article was concerned with 2 × 2 contingency tables, it provided the impetus for using the logarithmic transformation.

Plackett (1962) constructed a test of no second order interaction in an R × C × L contingency table that was an extension of the method implicitly given by Woolf. Plackett's method is based upon the analysis of the log frequencies of the observed table that used a transformation matrix which had rows that were orthogonal to each other. For the contingency table with R-rows, C-columns, and L-layers, Plackett formed (R-1) (C-1) linear combinations of the logarithmic frequencies for each layer. Each linear combination can be regarded as a vector having an asymptotic distribution that is multivariate normal. For the 2 × 2 × L contingency table, Plackett said that his analysis is computed more easily than Norton's because the evaluation of Plackett's statistic does not require estimates of cell frequencies. However, for the R × C × L contingency table, (L+1) square matrices of side (R-1) (C-1) must be inverted (Goodman, 1963).

In 1963, Goodman improved Plackett's method by using a matrix transformation consisting of rows that

were not orthogonal to each other. In this way only one matrix of side (R-1) (C-1) and L matrices of side u=min {R,C} had to be inverted. Goodman's test of no second order interaction is equivalent to Plackett's test and Plackett's test in turn is asymptotically equivalent to Bartlett's test and its extensions by Roy, Kastenbaum, and Lamphiear.

In the beginning of 1964, Goodman presented a paper which defined the r^{th} order interactions in a m-dimensional $d_1 \times d_2 \times \ldots \times d_m$ contingency table where r=0,1,2,...,(m-1). He derived methods for testing the hypothesis that any specified subset of these interactions are equal to zero. The tests presented were the generalizations of the methods derived by Plackett in 1962 and Goodman in 1963. In addition to the tests of various types of interactions, Goodman gave methods for obtaining simultaneous confidence intervals for these interactions.

Finally, Goodman (1964b) presented what he termed simple methods for analyzing second order interactions in contingency tables and for obtaining simultaneous confidence intervals used to estimate the magnitude of the second order interactions. The methods discussed in his paper reduced the number of matrices that must be inverted in the R × C × L contingency table. Since Goodman's tests are modifications of his earlier works and those of Plackett, once again

multivariate normal theory is used and some martrix inversion is required.

Loglinear Modeling/Iterative Proportional Fitting

In 1963, Birch rewrote Roy and Kastenbaum's definition of no second order interaction in terms of logarithms. $Ln(m_{ijk})$ can be written as a sum of parameters which represent eight effects (Birch, 1963). The eight effects parallel the concepts of linear modeling in the analysis of variance for continuous data. For the R × C × L contingency table the possible eight effects are: an overall mean effect, three main effects (effects for each classification), three first order effects (effects resulting from interactions between two of three attributes), and one second order effect (effect resulting from the interaction among the three attributes). When the model of no second order interaction is hypothesized, the second order parameter is set equal to zero. Birch went on to prove a number of theorems that dealt with the uniqueness of the maximum likelihood estimates of the eight parameters. Birch referred also to methods given by Kastenbaum, Lamphiear and Darroch for obtaining solutions to the maximumn likelihood equations of the cell expectations.

Bishop (1969) used a computational method for estimating cell expectations that was an adaptation of the Deming-Stephan algorithm (1940). The algorithm used a least squares adjustment to adjust cell entries so that the cell entries fit a new set of marginal totals but keep their original relationship to each other. With the Birch loglinear modeling concept and

the Deming-Stephan algorithm, Bishop was able to test various hypotheses of interaction in an R × C × L contingency table. Each hypothesis results in a model with one or more parameters set equal to zero. The model in turn determines a unique set of configurations that are sufficient for fitting the data to the model. A configuration is a table in which each entry is a sum of entries from rows, columns, or layers of the given contingency table.

Bishop introduced the concept of hierarchical models (Bishop, 1969). For a hierarchical model, parameters are omitted from the model only in descending order of dimensionality. For example, if U_{12} represents the first order interaction effect of R and C in an R × C × L contingency table, then U_1 (the R effect) and U_2 (the C effect) must also be present. For a nonhierarchical model, Fienberg (1977) suggested transforming the table so that the new model is hierarchical.

For some models, the cell estimates can be written directly from the marginal totals. These are called multiplicative models or models in which there exists closed form estimates of cell expectations (Bishop, 1969; Bishop, Fienberg, & Holland, 1975; Fienberg, 1977). For the R × C × L contingency table there is only one model that is not multiplicative. This is the model which hypothesizes no second order interaction. Consequently, the cell expectations must be found iteratively. Bishop, Fienberg, and Holland (1975) favor

iterative proportional fitting over the techniques of Roy and Kastenbaum and Goodman for two reasons. First, the Iterative Proportional Fitting method depends on only the sufficient configurations and no special provisions need to be made for sporadic cells with no observed values. Second, when closed form estimates exist, the Iterative proportional Fitting method gives the exact cell estimates in one cycle of iteration.

Once cell expectations are found, the goodness-of-fit of the model is checked by means of either X^2 or L^2 (see Equations 1 and 2).

Alternate Formulations of Second Order Interaction

Mood (1950) discussed briefly four testable hypotheses for a three-way contingency table. However, the four hypotheses were concerned only with mutual independence and tests of whether or not any one of the categories was independent of the other two. In other words, Mood ignored the hypothesis of no second order interaction.

In 1951, Lancaster derived a partitioning procedure for the total χ^2 so as to investigate the interactions of all orders in higher dimensional contingency tables. He claimed that for the $2 \times 2 \times 2$ table, the χ^2 for second order interaction was asymptotically equivalent to the χ^2 for second order interaction obtained by Bartlett (Lancaster, 1951). The Lancaster procedure defined the total χ^2 as a sum of χ^2 components that resulted from the various interactions

in the table.

In the same year, 1951, Simpson discussed Bartlett's definition of second order interaction in great detail. He accepted Bartlett's definition of second order interaction and pointed out that whatever function of cell probabilities is used to describe second order interaction, the function should be symmetrical with respect to the three methods of classification. In a footnote to Simpson's paper, the editor noted that Lancaster's χ^2 component for second order interaction was not equivalent necessarily to Bartlett's definition of second order interaction. In 1962, Plackett gave an example of a three-way contingency table which satisfied Bartlett's condition for second order interaction, but did not satisfy Lancaster's definition of second order interaction. Plackett accepted Bartlett's definition of second order interaction and developed a test of second order interaction which was based on an analysis of log frequencies. Later, other authors presented disclaimers of Lancaster's partitioning procedure (Lewis, 1962; Goodman, 1964; Lindley, 1964; Shaffner, 1971).

Mayo (1961) devised a graphical procedure which pictorially describes second order interaction in a $2 \times 2 \times 2$ contingency table. Mayo gave examples of six second order interaction patterns; but he acknowledged that there could be more than six patterns or that there could be sub patterns. The procedure consisted of sketching three lines which were

designated C_1, C_2, and T. C_1 represented the interaction for the first layer, C_2 represented the interaction for the second layer, and T represented the interaction line for the entire table. The lines were constructed from the two proportions of the cell frequencies in the rows while the column totals were used as bases. The relationship among the three lines determined the pattern used to describe the second order interaction.

Darroch (1962) presented what he termed were direct continuations of the works of Bartlett, Norton, Roy, Kastenbaum, and Lamphiear. If p_{ijk} represents the probability of an observation falling into the i,j,k cell of an R × C × L contingency table, $p_{i.k}$ represents a typical one way marginal sum of probabilities, and $p_{i..}$ represents a typical two-way marginal sum of probabilities, Darroch used Roy and Kastenbaum's symmetrical definition of no second order interaction:

$$p_{ijk} = p_{.jk}p_{i.k}p_{ij.}/(p_{i..}p_{.j.}p_{..k}) \quad (8)$$

to prove that his definition led to marginal constraints of the form:

$$p_{i..}p_{.j.} = \sum_k (p_{i.k}p_{.jk})/p_{..k} \text{ (Ku \& Kullback, 1968).} \quad (9)$$

Darroch defined a perfect contingency table as one in which the symmetrical definition of no second order interaction and the set of marginal constraints were satisfied. He called a contingency table imperfect if it was impossible to express the cell probabilities as simple functions of the

marginal probabilities.

Although the existence and uniqueness of a solution to Bartlett's equation (see Equation 4) was proven, the extension to the set of constraints developed by Roy and Kastenbaum (see Equation 7) was not proved until 1963 (Birch, 1963). Nevertheless, Darroch assumed the existence and uniqueness of a solution and eventually derived an iterative technique for estimating cell frequencies under the hypothesis of no second order interaction. For an $R \times C \times L$ contingency table, $RL + RC + CL$ equations must be solved iteratively when the table is not perfect. The number of equations which must be solved using the methods of Darroch is greater than the $(R-1)(C-1)(L-1)$ equations given by Kastenbaum and Lamphiear. However, Darroch claimed that his system was easier to solve than Kastenbaum and Lamphiear's system and his method converged more rapidly to the solution than did the Kastenbaum and Lamphiear method.

In 1963 Good investigated higher order interactions in contingency tables by means of the Principle of Maximum Enthropy. A brief explanation of this principle follows.

Suppose X is a random variable whose probability distribution is not completely given; but whose distribution is subject to some set of restraints. Of all possible distributions there will usually be one of maximum uncertainty. Maximum enthropy is merely another name for maximum uncertainty. The Principle of Maximum Enthropy states that the null

hypothesis to be used should be the one of maximum enthropy under the given constraints (Good, 1963).

Good used this principle to generate testable hypotheses and derived the general r^{th} order constraints necessary to test the hypothesis of no r^{th} order and higher order interactions in an m-dimensional contingency table with $r<m$. The constraints were generated by means of discrete Fourier transforms of the logarithms of the probabilities. For the $2 \times 2 \times 2$ contingency table, the Principle of Maximum Enthropy yields the same cubic equation as originally proposed by Bartlett (see Equation 5). However, when there are more than two levels of the categories, Good's solutions to the set of constraints are complex valued and difficult to interpret (Ku & Kullback, 1968). In order to solve Good's system of equations, the user would use either Bartlett's procedure or the iterative solution given by Darroch (Goodman, 1964).

Lindley (1964) looked at the analysis of contingency tables with respect to a Bayesian point of view. Lindley considered the observed cell frequencies, n_i's, as random variables having a multinomial distribution with corresponding cell probabilities, p_i's. He assumed that the prior distribution of the p_i's had density proportional to $(\Pi p_i)^{-1}$ with $p_i \geq 0$ and $\Sigma p_i = 1$. With this assumption, he expressed Bartlett's definition of no second order interaction in terms of logarithms and thereby was able to define a parmeter,

Ψ, as a linear contrast of the logarithms. Lindley claimed that Ψ was normally distributed. From concepts used in analysis of variance, Ψ could be expressed as the sum of seven parameters which represented the various effects in the contingency table. Lindley derived a test statistic for the hypothesis of no second order interaction that was based upon the differences between the logarithms of the observed values and the logarithms of the expected values.

Lindley then extended the concepts derived for the $2 \times 2 \times 2$ contingency table to the general $R \times C \times L$ contingency table. He noted that the calculations necessary to apply his test statistic are substantially reduced by use of Goodman's method which was derived in 1963.

Ku and Kullback (1968) investigated higher order interaction in a contingency table from the viewpoint of information theory. In earlier works on information theory for two-way contingency tables, the minimum discrimination information statistic (M.D.I. statistic):

$$2n\hat{I}(p:\pi) = 2\sum_{ij} n_{ij} Ln(n_{ij}/n\pi_{ij}) \qquad (10)$$

was used as the test statistic (Kullback, Kupperman, & Ku, 1962). In this statistic, n_{ij} is the observed frequency, π_{ij} is the probability of an observation falling into the i,j cell under the null hypothesis: $p = \pi$, and n is the sample size. Ku and Kullback extended earlier theorems on information theory from two-way tables to $R \times C \times L$ and higher dimensional tables. By letting

$p^*_{ijk} = a_i b_j c_k \pi_{ijk}$, then:

$$\text{Ln}(p^*_{ijk}/\pi_{ijk}) = \text{Ln}(a_{ij}) + \text{Ln}(b_{jk}) + \text{Ln}(c_{ik}) \qquad (11)$$

represented no second order interaction among the three methods of classification. a_{ij}, b_{jk}, and c_{ik} were functions of the given two-way marginal probabilities with $\sum_{ijk} a_{ij} b_{jk} c_{ik} \pi_{ijk} = 1$. Estimates of the p^*_{ijk}'s were found by means of the Deming-Stephan proportional fitting algorithm. After the estimates of the p^*_{ijk}'s were found, cell estimates, m_{ijk}'s, were determined and the M.D.I. statistic (see Equation 10) was used to test the hypothesis of no second order interaction. The M.D. I. statistic is asymptotically distributed as a χ^2 distribution on (R-1)(C-1)(L-1) degrees of freedom.

In the summary of their 1968 article, Ku and Kullback said that information theory was a unified approach to the analysis of multidimensional contingency tables because the principle of M.D.I. can be used to generate all hypotheses on interest when certain marginals are considered fixed. When, for example, two-way marginals are fixed, the p^*_{ijk}'s represent no second order interaction. Ku and Kullback showed that their approach was consistent with previous interpretations of second order interaction and they gave examples that showed how their statistic compared to other statistics used to test for no second order interaction.

The main advantage of using information theory is that it presents an additive analysis of the entire contingency table. Since hypotheses can be generated by this approach,

the entire table can be analyzed with respect to each generated hypothesis (Ku, Varner, & Kullback, 1971). Ku and Kullback proved that $I(p:\pi) = I(p:p^*) + I(p^*:\pi)$ is true for p^* computed by the proportional fitting method when p and p^* share the same marginal constraints (Ku & Kullback, 1974).

In 1969, Nagnur derived a set of tests for testing the hypothesis of no second order interaction in a three dimensional contingency table. The set consisted of a test of no second order interaction in a $2 \times 2 \times 2$ table, a test of no second order interaction in a $2 \times 2 \times L$ table, and a test of no second order interaction in an $R \times C \times L$ table. Nagnur uses the criterion for no second order interaction which was set down by Roy and Kastenbaum (see Equation 7). In a manner similar to Goodman's method, Nagnur defined the null hypothesis of no second order interaction in the form of linear contrasts in the logarithms of the theoretical cell probabilities. However, in order to use Nagnur's tests, one must still obtain estimates of cell probabilities. The estimated probabilities and the observed cell frequencies are substituted into the appropriate test statistic. The result of the test statistic is compared with the theoretical χ^2 value on $(R-1)(C-1)(L-1)$ degrees of freedom.

Gokhale (1971) derived an iterative procedure for analyzing loglinear models in multinomial experiments.

Gokhale's technique is similar to the iterative proportional fitting procedure mentioned earlier. The difference between the two methods is due to the sufficient statistics used to calculate cell expectations. In the Iterative Proportional Fitting method, the set of cell configurations is considered minimal whereas in Gokhale's technique there is no provision for minimal statistics. This means that the iterative proportional fitting technique will converge to cell estimates more rapidly than Gokhale's procedure.

Patil (1974) proposed a test of no second order interaction for an R × C × L contingency table that was an extension of a test proposed by Gart (1972) for 2 × C × L contingency tables. Gart's test was an extension of a test proposed by Zelen (1971) for a 2 × 2 × L contingency table. Patil accepted Roy and Kastenbaum's criterion for no second order interaction in an R × C × L contingency table (see Equation 7). Patil remarked that his statistic is comparable to Goodman's statistic. The two methods differ with respect to the data used to compute the statistics. Goodman set up contrasts using the natural logarithms of the cell observations whereas Patil used the actual cell observations.

Summary of Interpretations of Second Order Interaction

This concludes the review of various interpretations of second order interaction and various statistics used to test the hypothesis of no second order interaction in an R × C × L contingency table. A thread which seems to

wind through the discussion of second order interaction is the direct and indirect references to the criterion for no second order interaction that was given by Bartlett in 1935. As Ku and Kullback said in 1968 concerning Bartlett's definition of second order interaction:

> It is remarkable that his definition remains the preferred one to this date and the same hypothesis has been arrived at by others through different approaches (Ku & Kullback, 1968, p. 161).

The investigations into second order interaction in multidimensional contingency tables fall into two categories. The first category consists of those techniques used to obtain cell estimates. The second category is concerned with methods used to test hypotheses about interaction.

In the first category basically there are two ways of obtaining cell estimates. The methods of Bartlett-Fisher, Norton, Roy, Kastenbaum, Lamphiear, and Darroch involve iterative procedures for solving systems of equations. Goodman called this the BFNRKLD test (Goodman, 1964). The statistics that use some form of iterative proportional fitting comprise the second way of obtaining cell estimates. In this group belong the methods of Bishop, Fienberg, Holland, Haberman, Nagnur, Gokhale, and Birch.

In the second category, the methods analogous to Goodman's procedure exemplify the concerns of hypothesis testing in contingency tables. This category includes the works of Woolf, Plackett, Goodman, Good, Ku, Kullback, and Patil.

The Bartlett method and its extensions by Norton, Roy, Kastenbaum, and Lamphiear seem to be the standards used for comparisons by the later methods. The Iterative Proportional Fitting method as it is applied to loglinear modeling provides a completely different technique for obtaining cell estimates. The Goodman method encompasses those procedures concerned with hypothesis testing. Therefore, in the present investigation, the behavior of small samples in three dimensional contingency tables was investigated with respect to the behavior of the Bartlett, Goodman, and the Iterative Proportional Fitting methods on small samples.

Small Sample Studies

Shanawany (1936) computed the exact multinomial probabilities for two samples. Both samples consisted of three cells with cell probabilities of .3, .5, and .2. The first sample had a sample size of 10 and the second sample had a sample size of 20. For N = 10, there are 66 possible ways of distributing the 10 items among the three cells. For N = 20, there are 128 possible ways of distributing the 20 items among the three cells. Shanawany listed the 66 and 128 sequences, computed their exact multinomial probabilities, computed their χ^2 distribution values, and compared the exact probability, X^2, and χ^2 for each sequence. The results for both N = 10 and N = 20 indicated that the conclusions reached by using X^2 were for the most part the same as those

reached by using the exact multinomial distribution. Of course Shanawany's conclusions are applicable only to these two examples.

Freeman and Halton (1951) derived a method for analyzing contingency tables that had small sample sizes. They believed that χ^2 distribution was inaccurate when expected and observed values were small. Consequently, they hoped that their method would eliminate the inaccuracies which existed in the χ^2 distribution. They did not define the meaning of small sample but more than likely they followed the criteria of 5 to 10 for expected values (Lewis & Burke, 1949; Lewis & Burke, 1950).

The method is exact in the sense that the exact probability of obtaining a contingency table as probable as, or less probable than, the observed contingency table can be calculated. The test is carried out in the following manner.

All tables subject to the same marginal totals as the observed table are listed. Freeman and Halton used as an example a 2 × 3 table. For this example, 18 tables were listed.

The exact a priori probability of each table is computed. The method used to calculate the appropriate probabilities is determined by the hypothesis to be tested. In the 2 × 3 example, the null hypothesis was that the table is homogeneous.

The tables which have probabilities less than or equal to the probability of the observed table are noted. These

probabilities, including the probability of the observed table are summed up. If this total a priori is greater than the level of rejection (say, .05), then the null hypothesis is not rejected.

Freeman and Halton gave a second example that illustrated their technique for a 2 × 2 × 2 contingency table. As before, the null hypothesis was that the table is homogeneous. In this example, they gave a short cut method which reduced the number of tables that had to be listed. Even with the short cut, 43 tables were listed. Consequently the number of calculations could be quite large for higher dimensional tables or for R × C tables with more than two levels of classification for each attribute.

Wise (1963) derived a formula which he hoped would replace the usual goodness-of-fit test (see Equation 1) as the statistic which is usually approximated by the χ^2 distribution. Wise's test statistic is:

$$X'^2 = \Sigma(\text{observed} - \text{expected})^2/(\text{expected} + \tfrac{1}{2}). \qquad (12)$$

The derivation of X'^2 was based upon the assumption that the cell probabilities are nearly equal. With this assumption, he concluded that the expected frequencies need not be very large and that when computing likelihoods, it seemed more important that the observed frequencies should be equal as nearly as possible than that they should be large.

Wise gave two examples for samples of size eight. In the first example the exact probabilities were calculated

for a 1 × 4 table with equal cell probabilities. In the second example the exact probabilities were tabulated for a 1 × 3 table with cell probabilities of ½, ½, and ¼.

The results indicated that the χ^2 distribution is approximated better by X'^2 than by Pearson's X^2 (see Equation 1). However, Wise pointed out that there are many gaps to be filled before X'^2 would be accepted as a replacement for X^2. In particular, more studies had to be conducted for cases in which cell probabilities are not equal.

Lewontin and Felsenstein conducted a Monte Carlo investigation on contingency tables of the form 2 × n. The purpose of the study was to investigate the behavior of the Pearson statistic, X^2 (see Equation 1), with respect to the χ^2 distribution when the expectations of individual cells are small (Lewontin & Felsenstein, 1965).

The row classiciation, R = 2, was considered a Bernoulle experiment. The following notation was used: n (number of columns), a (number of total successes in the first row), N (total number of observations), $\bar{N}_i$ (mean number of observations in column i), N_i (the distribution of the N observations into the n columns), and a/N (proportion of successes). The following variables were manipulated: n, a/N, $\bar{N}_i$, and N_i and consequently N and the expected cell values were determined.

The results of the experiment showed that the probability of a Type I error given by X^2 (see Equation 1) was in general

conservative for 5 or more degrees of freedom even when cell expectations are very small in each cell. Lewontin and Felsenstein adopted a rule of operation that if cell expectations are 1 or more the X^2 is conservative at the 5%, 2%, and 1% level of significance and that for most cases even fractional expectations do not affect the test. Furthermore, they noted that for those cases of fractional expectations in which X^2 is nonconservative, the deviations of the true α values from those given by the χ^2 distribution were quite small.

Slakter (1966) conducted a Monte Carlo experiment on the adequacy of the χ^2 approximation to the usual goodness-of-fit statistic, X^2 (see Equation 1), and two modifications of X^2:

$$Y^2 = N(X^2 - k/N)/(N-1) \tag{13}$$

$$W^2 = X^2 - k/N \tag{14}$$

where N is the total sample size and k is the number of cells. In the study, expected cell frequencies were taken to be equal and this resulted in equal cell probabilities of 1/k. Under this assumption, X^2 simplifies to:

$$X^2 = (k/N)\Sigma n_i^2 - N \tag{15}$$

where n_i is the observed cell value of the i^{th} cell. Y^2 is a statistic proposed by Nass (1959) which provides for a correction for continuity when the expected frequencies are small but equal. W^2 is X^2 when the correction for continuity is applied.

Sampling distributions were constructed for all combinations of table size (10, 25, 50) and k values (10 to 250). Under these conditions, expected cell values ranged from 5 to .05. For each sampling distribution, the Monte Carlo procedure generated 10,000 random samples. Each random sample was treated by each of the three statistics. Slakter was interested primarily in Type I error and his results were analyzed with respect to how conservative the three statistics were at α's of .01, .05, and .10. The results seemed to indicate that X^2 was the best choice if an experimenter were interested in minimizing a Type I error. On the other hand, W^2 was more conservative than X^2 and there seemed to be no evidence that Y^2 provided better results than either X^2 or W^2. The overall conclusion reached was that X^2 was robust when expected frequencies were small but equal.

In a follow up to his 1966 study, Slakter (1968) did a Monte Carlo study on the accuracy of an approximation to the power of the goodness-of-fit statistic, X^2 (see Equation 1) with small but equal expected frequencies. In the study, N (sample size), k (number of cells), and α (level of rejection) were manipulated. There were three levels of N (10, 25, 50), four levels of k (10, 20, 31, 41), and two levels of α (.01, .05). In order to test for power, Slakter constructed nine alternative hypotheses so that their respective nominal powers were: .1, .2, .3, ..., .9. Under these constraints

there were 24 combinations of N, k, and α. Therefore, these 24 combinations in conjunction with the 9 alternative hypotheses resulted in a total of 216 testable hypotheses. The sampling procedure was repeated 1,000 times for each hypothesis and this resulted in expected values ranging from less than ¼ to 5.

The results seemed to indicate that power was related more to sample size than to the size of the expected frequencies. For samples in the range of 10 to 50, Slakter offered a crude working rule that if P_∞ represents the nominal power of P_α the actual power then: $P_\alpha = .8P_\infty$. When N is greater than 50, this rule seemed to underestimate the actual power. For N under 10, the value of P_α given by this rule seemed to be higher than the actual power.

Good, Gover, and Mitchell (1970) investigated the distributions of X^2 (see Equation 1) and:

$$\Lambda = 2NLn(k) - 2NLn(N) + 2n_iLn(n_i) \qquad (16)$$

for multinomial distributions. In Equation 16, N is the sample size, n_i is the observed cell frequency of the i^{th} cell, and k is the number of cells. They were interested in obtaining information concerning: the exact distributions of X^2 and Λ, the closeness of the χ^2 to X^2 and Λ, and the possibility of using the Poisson distribution as an approximation to the X^2 distribution.

Good, Gover, and Mitchell varied the sample size and the number of cells. There were various combinations of cell

numbers that ranged from 3 to 18 and table sizes that ranged from 3 to 28. The authors stopped at N = 28 because the number of calculations was dependent upon the number of ways of partitioning the sample size among the cells. Good, Gover, and Mitchell smoothed out the tail-area probabilities of the X^2 distributions by using a series of step functions. By smoothing out these tails, they were able to avoid using the correction formula for X^2 (see Equation 14).

The results showed that X^2 had a smoother distribution than Λ. In the closeness of the χ^2 to $X^2$and Λ, it appeared that X^2 sometimes is adequate for the equiprobable multinomial even when the cell expectations are as low as 1/3. However, Λ is not as adequately approximated by χ^2 as is X^2. The results that concerned χ^2 and the Poisson distribution indicated that between χ^2 and Poisson, either one can give reasonable estimates of X^2. For example: when k = 10 and N = 3, 4, or 5, the Poisson is better. When k = 10 and N = 6, 7, or 8, there is not much of a difference. When k = 10 and N = 10, χ^2 is better.

Craddock and Flood (1970) investigated the distribution of X^2 (see Equation 1) in R × C contingency tables of dimensions from 5 × 5 down to 3 × 2, 5 × 1, 4 × 1, and 3 × 1. For this study, which was a follow up of an earlier work on 3 × 3 contingency tables (Craddock, 1966), the average cell expectations were about equal and ranged from 1 to 5. For each contingency table, at least 10,000 values of X^2 were

found in order to determine satisfactorily the percentile values of each X^2 distribution. Table sizes ranged from N = 5RC to RC. For large N, the experimentally determined percentiles were always near those of the χ^2 distribution. As N decreased the results matched those of Craddock's 1966 study: the middle values of X^2 increased while the smaller and higher values of X^2 decreased. The changes in X^2 were small until the cell expectations became on the average less than 1. For smaller values of N, the results were questionable and consequently, Craddock and Flood suggested that the user look for other means of analyzing the contingency table.

Odoroff (1970) compared small sample properties of 12 goodness-of-fit tests for interaction in 2 × 2 × 2 and 3 × 2 × 2 contingency tables. The 12 tests were combinations of three tests (minimum logit chi-square, Pearson chi-square, and the likelihood ratio) and four methods of estimation (iterative maximum likelihood and three variations of the non-iterative minimum logit chi-square estimator). Odoroff investigated the exact probabilities by listing all distributions of the given sample or by generating the distributions by means of a Monte Carlo technique. In addition, he did a limited study on the power of the 12 statistics.

For the 2 × 2 × 2 table, all fixed cell margins, $n_{ij.}$, were set equal. There were 4 values of $n_{ij.}$ (5, 10, 15, 20) in combination with various fixed minimum cell expectations. The exact probabilities were computed and the levels of

rejection were taken to be .05 and .01. For the 2 × 2 × 2 tables with $n_{ij.} > 5$ and for the 3 × 2 × 2 tables it was too expensive to compute exact probabilities. In these cases, 2000 Monte Carlo samples were generated for each combination of $n_{ij.}$ and minimum cell expectancies. As in the 2 × 2 × 2 case, there were 4 values of $n_{ij.}$ (5, 10, 15, 20) and various sets of fixed minimum cell expectations. The investigation of power was restricted to the 2 × 2 × 2 case for $n_{ij.} = 10.\ 15.$ Six alternative hypotheses were tested.

Of the 12 test statistics, the minimum logit chi-square:

$$X^2 = \Sigma\Sigma \hat{W}_{ij}(\hat{Y}_{ij} - \tilde{Y}_{ij}) \qquad (17)$$

was the best choice and was the least sensitive to small cell expectations. In this statistic, $\tilde{Y}_{ij} = \tilde{\mu} + \tilde{\alpha}_i + \tilde{\beta}_j$ with $\tilde{\mu}$, $\tilde{\alpha}_i$, $\tilde{\beta}_j$ representing fitted parameters, $\hat{Y}_{ij} = 2 + 1/(n_{ij.} - \frac{1}{2})$, and $\hat{W}_{ij}^{-1} = 1/(n_{ij} + \frac{1}{2}) + 1/(z_{ij} + \frac{1}{2})$ with $z_{ij} = m_{ij} - n_{ij}$ for observed value n_{ij} and expected value m_{ij}. With respect to power, no test was more powerful than any other test.

Zahn and Roberts (1971) investigated the accuracy of the χ^2 distribution under the test statistic, X^2 (see Equation 1) when cell expectations were one. In addition, they compared X^2 to the "zeros" test proposed by David (1950).

Zahn and Roberts tabulated the exact distributions of X^2 for most table sizes in the range of N = 1 to N = 50. Since all cell expectations were one and since cell expectations must sum to N, this meant that there were N cells and X^2

simplified to:

$$X^2 = \Sigma n_i(n_i - 1) \qquad (18)$$

where n_i is the observed frequency in the i^{th} cell. In addition, $n_i(n_i - 1)$ is always even and so X^2 assumed only even values. After all partitions for a given N were formed, the probabilities for each partition were calculated and the exact cumulative distribution was found.

The results indicated that the χ^2 approximation by means of X^2 did much better in the right tail of the distribution than it did in the left tail of the distribution. This happened because X^2 is smoother in the right tail than in the left tail. In the left tail, exact values should be used rather than the χ^2 approximation. Finally, X^2 was superior to the zeros test of David.

McNamee (1973) did a Monte Carlo study on first and second order interaction in 2 × 2 × L contingency tables. The investigation focused on the question: Is the chi-square value for first and second order interaction dependent upon sample size and dimension? Is the table size or the individual layers the determining factor of robustness of the chi-square tests for first and second order interaction? How robust is the Bartlett method for calculating the chi-square for second order interaction? Four hundred random tables under the equiprobable distribution were generated for various combinations of layer size, L, and cell expectations. The proportionate number of times that a set of 400 tables was rejected

was compared to the theoretical .10, .05, .02, and .01 levels of significance.

The results of this study indicated that the chi-square tests were very robust and any error is less than the experimental error of the study. In addition, McNamee encountered a number of negative chi-square values due to the violation of an assumption in the Bartlett method that a solution to the cubic equation (see Equation 5) relies on the convergence of an infinite series. These tables were reanalyzed by interchanging the columns in each row. Of the 1,327 tables with negative chi-square values, all but four gave positive values after the columns were interchanged.

Tate and Hyer (1973) reviewed the results of two sampling experiments on the accuracy of X^2 (see Equation 1) that were conducted by Slakter (1966) and Roscoe and Byars (1971). These Monte Carlo experiments demonstrated that the X^2 statistic follows approximately the χ^2 distribution regardless of the cell expectations. However, Tate and Hyer were concerned more with the accuracy of the χ^2 distribution as an approximation to a single multinomial distribution. Tate and Hyer examined the results of an earlier study (Tate & Hyer, 1969) in which they noted that the most important source of inaccuracy in the approximation was the number of outcomes yielding the same X^2 and having different cumulative multinomial probabilities.

In their 1969 study, Tate and Hyer constructed 126

multinomial distributions with the number of cells, k, varying from 3 to 7 and the sample size varying so that cell expectations ranged from 1 to not fewer than 5. Also they constructed 36 distributions for k = 3 and N ranging from 4 to 12. After re-analyzing these results for the present article, Tate and Hyer concluded that the X^2 test seemed unsatisfactory when expectations are small. They recommended that not fewer than 20 expectations per cell be the criterion to follow in deciding whether or not the X^2 test is to be employed. Furthermore, since X^2 seemed to be untrustworthy for small expectations, studies on the power of the test have little value.

Margolin and Light (1974) did a small sample comparison of X^2 (see Equation 1) with Light and Margolin's C statistic and Kullback's minimum discrimination information statistic, $2\hat{I}$. For the I × J contingency table,

$$C = (N - 1)(I - 1)BSS/TSS \qquad (19)$$

where: $BSS = \frac{1}{2}\sum\sum(1/n_{.j})(n_{ij}-(n_{i.}n_{.j})/N)^2$ and

$TSS = N/2 - (\frac{1}{2}N\sum n_{i.}^2)$ with sample size of N, marginal totals of $n_{i.}$ and $n_{.j}$, and n_{ij}= observed frequency. The minimum discriminant information statistic is:

$$2\hat{I} = 2\sum NLn(N) + \sum\sum n_{ij}Ln(n_{ij}) - \sum n_{i.}Ln(n_{i.}) - \sum n_{.j}Ln(n_{.j}) \qquad (20)$$

In this study Margolin and Light tested the behavior of these three statistics under the null hypothesis of independence. The study was restricted to 3 × 2 tables with two fixed table sizes of 10 and 20. Furthermore, they let

$n_{.1} = n_{.2} = N/2$. Under the hypothesis of independence, this forced $p_{ij} = p_i$ for all i and j where p_{ij} is the probability of an occurence falling into the ij^{th} cell. With these assumptions, Margolin and Light had to specify only two of the three row probabilities. They computed exact probabilities for 22 combinations of 2 sample sizes and 11 sets of underlying probabilities.

The results indicated that C is better approximated by χ^2 than is $2\hat{I}$. X^2 was more conservative than $2\hat{I}$ and consequently $2\hat{I}$ should be avoided in testing independence in tables with small samples.

Korducki (1977) did a limited Monte Carlo study on the behavior of the Pearson X^2 statistic (see Equation 1) and the likelihood ratio statistic (see Equation 2) when the cell frequencies were small. The Monte Carlo simulation was done on 3 × 3 × 3 and 4 × 4 × 4 contingency tables. The null hypothesis was no second order interaction.

Korducki used loglinear modeling concepts in which he fixed one two-way margin. Parameters were chosen so that there was no second order interaction in the model and so that minimum cell expectations were about .9, 1.8, and 3.6. The sample sizes were 200 and 400 for the 3 × 3 × 3 contingency tables and 400 and 800 for the 4 × 4 × 4 contingency tables. For each set of parameters 1000 random samples were drawn according to the probabilities determined by the set of parameters. The level of significance was .10.

The results suggested that X^2 did better than L^2 for the smaller minimum cell expectations (expectations as low as .9 for samples of size 200). L^2 values were generally higher than X^2 values. L^2 seemed to behave quite well when the cell expectations were 3.6 in samples of size 400 for $3 \times 3 \times 3$ tables and for samples of size 800 in $4 \times 4 \times 4$ tables.

Larntz (1978) examined small sample properties of three goodness-of-fit statistics: X^2 (see Equation 1), L^2 (see Equation 2), and

$$T^2 = \Sigma(\text{observed}^{\frac{1}{2}} + (\text{observed}+1)^{\frac{1}{2}} - (4\text{expected}+1)^{\frac{1}{2}})^2 \quad (21)$$

Five models were used to make the comparisons. Model 1 was the usual multinomial goodness-of-fit for a given set of probabilities. Model 2 was a parametric model that arises from data in a problem-solving experiment. Model 3 involved the test of no association in an $R \times C$ contingency table with both margins fixed. Model 4 was based upon quasi-independence in an incomplete two-way table. Model 5 dealt with the hypothesis of no second order interaction in a $3 \times 3 \times 3$ contingency table. The question Larntz asked was: For small samples, which of the three statistics is best approximated by the χ^2 distribution?

Monte Carlo techniques were used for all but the second model. In Model 2, 2,025 possible outcomes were enumerated by the computer and so the exact distributions of X^2, L^2, and T^2 were found. The level of significance adopted

for all models was .05.

The results indicated that L^2 and T^2 gave exact values in excess of the .05 level for moderate expected values in the range of 1.5 to 4.0. The Pearson statistic, X^2, reached exact values that were very close to the .05 level of significance.

Miller (1979) did a Monte Carlo study on the behavior of the Pearson, X^2 (see Equation 1) and L^2 (see Equation 2) for one sample tests with small sample sizes. Miller varied the number of cells from k = 4 to k = 8 and the expected cell frequencies (3, 5, 10). This resulted in one sample sizes of 12 to 80. The primary goal of the study was to decide which statistic is better for small sample sizes. Miller generated 1000 random tables for each combination of cell number and expected cell frequency. The random numbers were generated from gamma distributions. This was done in order to verify Siegel's claim that nonparametric techniques are distribution-free. The level of significance was .10.

The results indicated that for k = 4 and expectancies of 10 and for k=5 and expectancies of 10, X^2 and L^2 behave quite well when compared to χ^2. Miller stated that for k=4 and expected values of 5 and for k = 6 and expected values of 10, L^2 gave a better fit to the samples χ^2. However, Miller said that the obvious disadvantage of L^2 for small samples is the number of statistics that are indeterminant. He claimed that L^2 is indeterminant because of the factor

Ln(0) that occurs in L^2 when the observed value is 0. However, 0Ln(0) is 0. Therefore, Miller's results need to be re-examined.

In 1980, Cox and Plackett made a study of small samples for 2^k, 2 × 2 × L, and 3 × 3 × 3 contingency tables. They investigated three ways of improving asymptotic methods that are used to make inferences in contingency tables. The three ways are: the use of conditional inference rather than the use of maximum likelihood estimators for approximating cell frequencies, exact distributions obtained by listing all tables for a given marginal total, and Monte Carlo techniques for tables where exact numeration is not feasible.

Cox and Plackett reached the following conclusions: First, conditional inference provides better approximations for cell expectancies than does maximum likelihood estimators. Second, exact distributions are useful for only a restricted class of hypotheses (e.g. fixed marginal totals). Third, although the use of conditional inference provides more accurate approximations for cell expectations than does unconditional maximum likelihood estimates, the amount of improvement has little effect on asymptotic methods of inference. For example, X^2 (see Equation 1) is closer to χ^2 than is L^2 (see Equation 2).

This review of small sample studies showed that for the most part the investigations of small samples dealt with the

behavior of various test statistics used to approximate χ^2. Except for Odoroff (1970), McNamee (1973), Korducki (1977), and Larntz (1978), most Monte Carlo small sample studies did not investigate second order interaction in higher dimensional tables. For the most part, the small sample studies seem to indicate that the approximations to χ^2 are robust when the cell expectations are small.

Recapitulation

The Bartlett, Goodman, and Iterative Proportional Fitting methods represent three distinct approaches to the testing of no second order interaction in three dimensional contingency tables. The Bartlett method summarizes those procedures which require an iterative solution to a set of equations. Iterative proportional fitting with respect to loglinear models estimates cells expectations without solving a set of equations. The Goodman method does not compute cell expectations; but rather is concerned with hypothesis testing.

The small sample studies were mostly concerned with robustness of different statistics used to approximate χ^2. No one has determined the robustness of these different methods for testing second order interaction when the sample sizes are small. McNamee (1973) indicated that the Bartlett method was robust for $2 \times 2 \times L$ contingency tables with small cell expectations. However, only 400 samples were generated randomly for any one combination of table size

and number of layers. In addition, McNamee did not investigate contingency tables that had zeros for cell frequencies and did not investigate tables that had more than two rows and two columns.

Therefore, the purpose of the present study was to see how robust the Bartlett, Goodman, and Iterative Proportional Fitting methods are for small samples. Specifically, the study investigated these methods under various combinations of sample size (20, 40, 60, 80, 100), dimension ($2 \times 2 \times 2$, $2 \times 2 \times 3$, $3 \times 3 \times 3$), and various underlying multinomial sampling distributions (four distributions for each dimension).

CHAPTER III

METHOD

Introduction

This chapter describes the methodology used in the present investigation of the behavior of the Bartlett method, the Goodman method, and the Iterative Proportional Fitting method as applied to loglinear models (from now on called "the Iterative Proportional Fitting method") when testing for second order interaction in three dimensional contingency tables that have small sample sizes. The chapter consists of six parts.

Part one defines the independent (methods, dimension, sample size, sampling distribution) and dependent variables (chi-square statistics of the methods) that were used in the present study, and the statistical hypotheses that were tested in terms of these independent and dependent variables.

Part two gives the rationale for doing a Type I error study as opposed to doing a Type II error study with respect to the number of times the null hypothesis, no second order interaction is rejected.

Part three discusses the reason for using a Monte Carlo technique to approximate rejection rates of the null hypothesis rather than to compute the exact probabilities

of all possible contingency tables for a given sample size under a given theoretical distribution.

Part four contains detailed discussions of the various procedures utilized in this study. These are: 1) the methods (Bartlett, Goodman, Iterative Proportional Fitting), 2) the rationale for dimensions (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2), 3) the rationale for sample sizes (20, 40, 60, 80, 100), 4) the rationale for sampling distributions (e.g. in a 3 × 3 × 3: 2:3:5, 2:3:5, 2:3:5 for row, column, layer, respectively), 5) the rationale for the number of tables generated (called the number of iterations) for a given set of constraints (1,825 tables), 6) the methodology employed to handle cells with zero frequencies, 7) the procedure used to handle potential negative chi-square statistics, 8) the methodology used to resolve computer-related problems (e.g. the problem of nearly singular matrices), and 9) the design and statistical analyses employed to test the hypotheses given in part one.

Statement of Hypotheses

In the present study the independent variables that where controlled were: method (Bartlett, Goodman, Iterative Proportional Fitting), sample size (20, 40, 60, 80, 100), dimension (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2), and sampling distribution (four distributions per dimension). The dependent variable was the rejection rate of the null

hypothesis, H_o: no second order interaction in the $R \times C \times L$ contingency table. The rejection rates were compared to the theoretical 5% point in the upper tail of the χ^2 distribution. Five sample sizes, three dimensions, and four sampling distributions resulted in 60 combinations of 1,825 randomly generated contingency tables. Each table was tested for second order interaction by each method. The independent and dependent variables led to the following hypotheses:

Hypothesis 1: The rejection rates of the methods (Bartlett, Iterative Proportional Fitting, Goodman) are the same. For this hypothesis, the overall rejection rates of the methods were analyzed while ignoring sample size (20, 40, 60, 80, 100), dimension ($3 \times 3 \times 3$, $2 \times 2 \times 3$, $2 \times 2 \times 2$), and sampling distribution (four per dimension).

Hypothesis 2: For a given sample size, the rejection rates of the methods are the same. For this hypothesis, the rejection rates of the methods were analyzed with respect to a given sample size.

Hypothesis 3: For a given dimension, the rejection rates of the methods are the same. For this hypothesis, the rejection rates of the methods were analyzed with respect to a given dimension.

Hypothesis 4: For a given sample size and dimension, the rejection rates of the methods are the same. For this

hypothesis, the rejection rates of the methods were analyzed with respect to a given combination of sample size and dimension.

Hypothesis 5: For a given dimension and sampling distribution, the rejection rates of the methods are the same. For this hypothesis, the rejection rates of the methods were analyzed with respect to a given combination of dimension and sampling distribution.

Hypothesis 6: For a given sample size, dimension, and sampling distribution, the rejection rates of the methods are the same. For this hypothesis, the rejection rates of the methods were analyzed with respect to a given combination of sample size, dimension, and sampling distribution.

It should be noted that not all combinations of the independent variables are meaningful. Each dimension has four sampling distributions, but the sampling distributions are different for each dimension. This is necessarily so because the number of rows, columns, and layers are different for each dimension. Therefore, the sampling distributions were investigated only within a given dimension.

Rationale for a Type I Study

When inferences are made about population parameters, an error can be made in two ways. First, for a given level of significance, the null hypothesis may be rejected when in fact the null hypothesis is true. This is called

a Type I error, α. Second, the null hypothesis may be accepted when it should be rejected. This is called a Type II error, β. A Type II error is related to the power of a statistical test by the relationship: power=1-β. Power refers to the strength of a test of significance to distinguish between the null hypothesis and a given alternative hypothesis.

The present study was concerned with a Type I error. The investigation centered on the rejection rates of the null hypothesis, H_o: no second order interaction in the R × C × L contingency table. If combinations of method, sample size, dimensionality, and sampling distribution have no affect on H_o, then a certain percentage of the computed statistics should fall in the rejection range. For example, if H_o is true, then for the χ^2 distribution with 1 degree of freedom, by chance alone values of a robust statistic should exceed 3.84 (the tabled value of the χ^2 distribution) only 5% of the time.

To investigate the power of a statistical test (which is essentially the same as investigating a Type II error), it is necessary to specify an exact value for the alternative hypothesis. Therefore, power must be demonstrated over a wide range of alternative hypotheses (e.g. see Slakter, 1968).

For the present study, the decision was made to investigate Type I error rather than Type II error because

an investigation of Type II error is of little value when the test statistic is untrustworthy (Tate & Hyer, 1973). In the present study, the Type I error was set at 5%.

Rationale for Using a Monte Carlo Technique

The ideal test of the robustness of a statistic is to calculate the exact probability of obtaining a result as extreme as the obtained result under the condition that the null hypothesis is true. With respect to contingency tables, this means that all contingency tables for a given sample size be enumerated, the respective probabilities for their occurrences under the null hypothesis be calculated, and the probability of obtaining a contingency table as extreme as the given table be determined. Then the obtained exact probability is compared to the probability found by the statistic to determine if the statistic is robust. The use of this method, however, presents a practical problem in the enumeration of the contingency tables. The amount of computation is a function of the number of cells and the sample size. Therefore, when the number of cells exceeds 18 or the sample size exceeds 28, then the number of calculations is prohibitive (Good, Gover, & Mitchell, 1970; Larntz, 1978).

An alternative approach to the investigation of the robustness of a statistic is to use a Monte Carlo technique. With respect to contingency tables, the Monte Carlo technique randomly generates a large number of

contingency tables (anywhere from 1,000 to 10,000) under a given set of constraints. Each table is tested by the statistic and a count is kept of the number of times the null hypothesis is rejected. The percentage of rejection found by the Monte Carlo technique is compared to the level of rejection. The decision then is made whether or not the discrepancy between the two percentages is tolerable under the hypothesis that the statistic is robust.

Since the present study used sample sizes of 20 to 100 and contingency tables having 8, 12, and 27 cells (i.e. $2 \times 2 \times 2$, $2 \times 2 \times 3$, $3 \times 3 \times 3$), it was more practical to use a Monte Carlo approach to investigate the robustness of the Bartlett, Goodman, and Iterative Proportional Fitting methods for small sample sizes than to enumerate all possible contingency tables.

Procedures

Introduction

This part of the Method chapter contains a discussion of the methods used to test for second order interaction (Bartlett, Goodman, Iterative Proportional Fitting) and a discussion of the methods used to determine the levels of the independent variables (sample size, dimension, sampling distribution). In addition there is a discussion of the procedure used to determine the number of tables to be generated (1,825), the procedure used to construct a contingency table, the methodology used in

tables that had cells with random zero frequencies, the methodology used for negative chi-square values, and the procedures used to solve computer-related problems (e.g. the problem of nearly singular matrices).

Test for Second Order Interaction

The Bartlett Method

The Bartlett method encompasses the method used to test for second order interaction in 2 × 2 × 2 contingency tables that was developed by Bartlett and the methods derived by Norton for 2 × 2 × L contingency tables and by Kastenbaum and Lamphiear for R × C × L contingency tables (Bartlett, 1935; Norton, 1945; Kastenbaum & Lamphiear, 1959). Therefore, the explanation of the Bartlett method will be given for the R × C × L contingency tables as it was derived by Kastenbaum and Lamphiear.

H_o, the hypothesis of no second order interaction in an R × C × L contingency table, was stated in Equation 7:

$$(p_{RCL}p_{ijL})/(p_{iCL}p_{RjL}) = (p_{RCk}p_{ijk})/(p_{iCk}p_{Rjk}),$$

for i=1,2,...,(R-1); j=1,2,...,(C-1); and k=1,2,...,(L-1). The p_{ijk}s are the parameters of the multinomial distribution

$$\phi=\{N!/(\Pi\Pi\Pi n_{ijk})\}\Pi\Pi\Pi p_{ijk}^{n_{ijk}}, \qquad (21)$$

for $\Sigma\Sigma\Sigma n_{ijk}=N$ and $\Sigma\Sigma\Sigma p_{ijk}=1$.

In Equation 21, n_{ijk} is the observed frequency and p_{ijk} is the probability of the ijk-cell and N is the sample

size.

Under H_o, estimates of the p_{ijk}s can be found by solving the following system of simultaneous third-degree equations in x_{ijk}:

$$(n_{RCL} - \sum_{i}^{R-1} \sum_{j}^{C-1} x_{ijL})(n_{ijL} - x_{ijL}) / \{(n_{iCL} + \sum_{j}^{C-1} x_{ijk})(n_{RjL} + \sum_{i}^{R-1} x_{ijL})\} = (n_{RCk} - \sum_{i}^{R-1} \sum_{j}^{C-1} x_{ijk})(n_{ijk} - x_{ijk}) / \{(n_{iCk} + \sum_{j}^{C-1} x_{ijk})(n_{Rjk} + \sum_{i}^{R-1} x_{ijk})\} \quad (22)$$

where $i=1,2,\ldots,(R-1)$; $j=1,2,\ldots,(C-1)$; $k=1,2,\ldots,L$

and where

$$x_{ijL} = -\sum_{k}^{L-1} x_{ijk}, \text{ or } \sum_{k}^{L} x_{ijk} = 0. \quad (23)$$

Since Equation 22 represents simultaneous systems of cubic equations in x_{ijk}, Kastenbaum and Lamphiear found the solution by means of the following iterative procedure.

Let all $x_{ijk}=0$, except for one set, say, x_{11k}. The general term of Equation 22 can be written

$$a_{11k} = (n_{RCk} - x_{11k})(n_{11k} - x_{11k}) / \{(n_{1Ck} + x_{11k})(n_{R1k} + x_{11k})\} \quad (24)$$

Equation 24 may be written

$$a_{11k} = (n_{11k} n_{RCk}) / (n_{1Ck} n_{R1k}) (\{1 - x_{11k}/n_{11k}\}\{1 - x_{11k}/n_{RCk}\} / \{(1 + x_{11k}/n_{1Ck})(1 + x_{11k}/n_{R1k})\}) \quad (25)$$

Equation 25 may be approximated by

$$a_{11k}=\{n_{11k}n_{RCk}/(n_{1Ck}N_{R1k})\}\{1-(1/n_{11k}+1/n_{RCk})x_{11k}\}/\{1+(1/n_{1Ck}+1/n_{R1k})x_{11k}\} \quad (26)$$

By expanding the terms in the denominator of Equation 26 which involve x_{11k}, the general term of Equation 22 can be written

$$a_{11k}=(n_{11k}n_{RCk})/(n_{1Ck}n_{R1k})\{1-x_{11k}/n_{11k}-x_{11k}/n_{1Ck}-x_{11k}/n_{R1k}/n_{RCk}\} \quad (27)$$

Letting

$$b_{11k}=(n_{1Ck}n_{R1k})/(n_{RCk}n_{11k}) \quad (28)$$

and

$$1/c_{11k}=1/n_{11k}+1/n_{1Ck}+1/n_{R1k}+1/n_{RCk} \quad (29)$$

Equation 27 becomes

$$a_{11k}=(1/b_{11k})(1-x_{11k}/c_{11k}) \quad (30)$$

From Equation 22, the a_{11k}s must be equal for all k=1,2,...,L. Therefore, equate a_{11k} and a_{11h} and solve for x_{11h}:

$$x_{11h}=c_{11h}(1-b_{11h}/b_{11k})(1-x_{11k}/c_{11k}) \quad (31)$$

But: $\sum_{h}^{L} x_{iih}=0$ (see Equation 23) and therefore

$$0=b_{11h}\sum_{h}^{L}c_{11h}-\sum_{h}^{L}c_{11h}b_{11h}+(x_{11k}/c_{11k})\sum_{h}^{L}c_{11h}b_{11h} \quad (32)$$

Solving Equation 32 for x_{11k}, then x_{11k} can be approximated by

$$x_{11k}^{(1)}=c_{11k}^{(1)}(1-d_{11}^{(1)}b_{11k}^{(1)}), \quad (33)$$

where the superscript (α) refers to the αth correction,

$$\sum_{\alpha}^{m} x_{ijk}^{(\alpha)} = x_{ijk} \tag{34}$$

where m is the number of iterations necessary to make $x_{ijk}^{(m)}=0$ with the desired accuracy and

$$d_{11}=(\sum_{k}^{L} c_{11k})/(\sum_{k}^{L} b_{11k}c_{11k}) \tag{35}$$

Equation 33 provides first approximations for the x_{ijk}s. These values of $x_{ijk}^{(1)}$ are either added or subtracted from the observed cell frequencies with which they are associated according to Equation 22. After this set of corrections has been applied, the iteration continues with the next set of equations involving, say, x_{12k}. Set all $x_{ijk}=0$, except for x_{12k}, solve for $x_{12k}^{(1)}$, apply these corrections to the appropriate cell frequencies, and continue this procedure for all $x_{ijk}^{(1)}$ with i=1,2,...,(R-1); j=1,2,...,(C-1). When all the first corrections have been determined and applied, the iteration begins again with $x_{11k}^{(2)}$ and continues until all $x_{ijk}^{(m)}=0$ with the desired accuracy. Five figure accuracy is sufficient for the x_{ijk}s (Norton, 1945). In the present study, the iterative procedure continued until the change in successive approximations of all x_{ijk}s was less than 1.0×10^{-6}.

In proceeding from Equation 25 to Equation 33, the $(1+x_{ijk}/n_{ijk})^{-1}$s (see Equation 25) are approximated by the first two terms of the series

$$1-x_{ijk}/n_{ijk}+(x_{ijk}/n_{ijk})^2-(x_{ijk}/n_{ijk})^3+... \quad (36)$$

This approximation holds provided $x_{ijk}/n_{ijk}<1$. Therefore, it may be necessary to estimate first corrections before using Equation 33. This is always the case for n_{ijk}s that are zero (Norton, 1945; Kastenbaum & Lamphiear, 1959). In the present study, when an $n_{ijk}$was zero, first corrections were found by means of the adjustment procedure recommended by Norton (1945). An explanation of this adjustment procedure is given in the Method chapter under the topical heading Procedures Used for Random Zeros.

After the cell estimates are found, let

$$\mu_{RCL}=\sum^{R-1}\sum^{C-1}\sum^{L-1} x_{ijk}', \qquad \mu_{RCk}=\sum^{R-1}\sum^{C-1} x_{ijk},$$

$$\mu_{iCL}=\sum^{C-1}\sum^{L-1} x_{ijk}', \qquad \mu_{iCk}=\sum^{C-1} x_{ijk}',$$

$$\mu_{RjL}=\sum^{R-1}\sum^{L-1} x_{ijk}', \qquad \mu_{Rjk}=\sum^{R-1} x_{ijk},$$

$$\mu_{ijL}=\sum^{L-1} x_{ijk}', \qquad \mu_{ijk}=x_{ijk} \quad (37)$$

Then the statistic

$$X^2=\sum^{R}\sum^{C}\sum^{L}\mu_{ijk}^2/(n_{ijk}+\nu_{ijk}\mu_{ijk}) \quad (38)$$

is distributed approximately as a χ^2 distribution on (R-1)(C-1)(L-1) degrees of freedom where

ν_{ijk}= 1 if i,j,k = R, C, L or if any two of the subscripts differ from R, C, L and

ν_{ijk}= -1 if only one or all three subscripts differ from R, C, L (Kastenbaum & Lamphiear, 1959). Equation 38

provides a statistic for testing H_o of Equation 7.

Goodman Method

The Goodman method is related to methods for the analysis of variance of observations in a two-way layout with unequal numbers in the cells (Goodman, 1964b). This procedure can be applied to the case where a random sample of size N is drawn from the three-way population in which the distribution of the observations in the table is multinomial. In addition, this method can be applied to the case in which the layer marginal frequencies are fixed and a random sample is drawn from the two-way population table corresponding to the kth layer of the R × C × L contingency table. This distribution of the observations in the two-way table, which corresponds to each layer, is multinomial. Finally, this procedure can be applied to cases in which the row (or column) marginal frequencies are fixed or where the layer by row marginal frequencies or layer by column marginal frequencies are fixed.

Let π_{ijk} be the probability that an observation will fall in the ith row, jth column, kth layer of a three-way table and let n_{ijk} denote the corresponding frequency in a sample of total size N. Then $\Sigma\pi_{ijk}=1$ and $\Sigma n_{ijk}=N$. Let θ_{ijk} denote the conditional probability that an observation will fall in the ith row and jth column given that it is in the kth layer. Then $\theta_{ijk}=\pi_{ijk}/\pi_{..k}$ where $\pi_{..k}=\Sigma_{ij}\pi_{ijk}$ and

$\sum_{ij} \theta_{ijk}=1$ for each value of k.

For the R × C × L table, the two-way population table which corresponds to the kth layer is an R × C contingency table and the first order interaction in the R × C table can be measured by the (R-1)(C-1) values:

$$\Delta_{ijk}=\theta_{ijk}\theta_{IJk}/\theta_{iJk}\theta_{Ijk}, \tag{39}$$

for i=1,2,...,(R-1) and j=1,2,...,(C-1) or by some function of the Δ_{ijk}. Each R × C table can be used to form (R-1)(C-1) 2 × 2 tables, where each 2 × 2 table so formed consists of the four cells in the ith row and Rth row (i=1,2,...,(R-1)) which are also in the jth column and Cth column of the R × C table with j=1,2,...,(C-1). The Δ_{ijk} measures the first order interaction in the 2 × 2 tables formed from the R × C table corresponding to the kth layer. The hypothesis H_o, no second order interaction in the R × C × L table is given by

$$H_o: \quad \Delta_{ij1}=\Delta_{ij2}=\ldots=\Delta_{ijL}$$

for i=1,2,...,(R-1) and j=1,2,...,(C-1). Letting $Ln(\Delta_{ijk})$ be denoted by Γ_{ijk}, then H_o can be written

$$H_o: \quad \Gamma_{ij1}=\Gamma_{ij2}=\ldots=\Gamma_{ijL}, \tag{40}$$

for i=1,2,...,(R-1) and j=1,2,...,(C-1).

Denoting the column vector $\{\Gamma_{i1k},\Gamma_{i2k},\ldots,\Gamma_{i,C-1,k}\}$ by $\Gamma_{i.k}$ and the column vector $\{\Gamma_{1.k},\Gamma_{2.k},\ldots,\Gamma_{R-1,.,k}\}$ by $\Gamma_{..k}$, H_o specifies that $\Gamma_{..k}$ is the same for k=1,2,...,L. For $R\geq 2$, $\Gamma_{..k}$ is a column vector with (R-1)(C-1) entries,

and the methods of multivariate analysis can be applied to test whether or not the L column vectors $\Gamma_{..k}$ (k=1,2,...,L) are equal. Δ_{ijk} is estimated by

$$d_{ijk}=(n_{ijk}n_{RCk})/(n_{iCk}n_{Rjk}), \tag{41}$$

Γ_{ijk} by

$$g_{ijk}=Ln(d_{ijk})=Ln(n_{ijk})+Ln(n_{RCk})-Ln(n_{iCk})-Ln(n_{Rjk}), \tag{42}$$

$\Gamma_{i.k}$ by

$$g_{i.k}=\{g_{il1}, g_{i2k},\cdots,g_{i,C-1,k}\}, \tag{43}$$

and finally, $\Gamma_{..k}$ by

$$g_{..k}=\{g_{1.k}, g_{2.k},\cdots,g_{R-1,.,k}\} \tag{44}$$

The asymptotic distribution of the vector $g_{..k}$ is multivariate normal, and let $T^{(k)}_{..}$ denote the variance-covariance matrix (Amick & Walberg, 1975) of $g_{..k}$. The matrix $T^{(k)}_{..}$, a matrix with (R-1)(C-1) rows and columns, consistently can be estimated by a matrix $U^{(k)}_{..}$ based upon the observed data. From multivariate normal theory, when H_o is true, the statistic

$$Y^2=\sum^{L}(g_{..k}- g_{..})'M^{(k)}_{..}(g_{..k}- g_{..}) \tag{45}$$

$$Y^2=\sum^{L}(g'_{..k}M^{(k)}_{..} g_{..k}) - \tilde{g}'_{..}Q_{..}\tilde{g}_{..} \tag{46}$$

is distributed asymptotically as a χ^2 distribution with (R-1)(C-1)(L-1) degrees of freedom according to the following definitions with ' representing the transpose of a given matrix.

$M^{(k)}_{..}$ is the inverse of $U^{(k)}_{..}$ (47)

$Q_{..}$ is the inverse of $\sum_{L}^{(k)} M_{..}$ (48)

$$g_{..} = Q_{..}\tilde{g}_{..} \tag{49}$$

$$\tilde{g}_{..} = \sum_{L}^{(k)} M_{..} g_{..k} \tag{50}$$

$U_{..}^{(k)}$ is a matrix of side (R-1)(C-1). However, it is not necessary to invert $U_{..}^{(k)}$. $M_{..}^{(k)}$ can be calculated more directly as a matrix of side R-1 where the entry $\mu_{ih}^{(k)}$ in the ith row and hth column (i,h=1,2,...,(R-1)) of $M_{..}^{(k)}$ is a matrix of side C-1 defined in the following manner.

$$\mu_{ih} = -B_{i}^{(k)} D^{(k)} B_{h}^{(k)}, \quad i \neq h \tag{51}$$

$$\mu_{ii} = B_{i}^{(k)} - B_{i}^{(k)} D^{(k)} B_{i}^{(k)}, \quad i=1,2,...,(R-1) \tag{52}$$

$B_{i}^{(k)}$ is a matrix of side C-1 where the entry $\beta_{jh}^{(ik)}$ in the jth row and hth column (j,h=1,2,...,(C-1)) of $B_{i}^{(k)}$ is

$$\beta_{jj}^{(ik)} = \beta_{j}^{(ik)} (1-\beta_{j}^{(ik)}/\beta^{(ik)}), \quad j=1,2,...,(J-1) \tag{53}$$

$$\beta_{jh}^{(ik)} = -\beta_{j}^{(ik)} \beta_{h}^{(ik)} / \beta^{(ik)}, \quad j \neq h, \tag{54}$$

with

$$\beta_{j}^{(ik)} = n_{ijk}, \quad \beta_{J}^{(ik)} = n_{iJk}, \quad \beta^{(ik)} = \sum_{j}^{J} n_{j}^{(ik)}, \tag{55}$$

and $D^{(k)}$ is the inverse of $\sum_{I}^{(k)} B_{i}$.

It should be noted that in Equation 42 it is necessary to take the natural logarithm of the quotients formed

in Equation 41. If any observed cell frequency is zero, then some of the equations defined by Equation 42 will be undefined. The procedure used to handle this problem is given in the Method chapter under the topical heading Procedures Used for Random Zeros

The Iterative Proportional Fitting Method

The Iterative Proportional Fitting method for estimating cell frequencies of a contingency table has been used extensively in loglinear analysis of contingency tables (Mosteller, 1968; Bishop, 1969; Goodman, 1970; Ku & Kullback, 1974; Bishop, Fienberg, & Holland, 1975; Reynolds, 1977; Haberman, 1979). To see why use of this procedure is preferred over the Bartlett method, it would be beneficial first to discuss the loglinear approach to contingency table analysis before discussing the iterative proportional fitting procedure. To provide a basis for a natural transition to R × C and R × C × L contingency tables, the discussion begins with a detailed explanation of loglinear analysis in 2 × 2 contingency tables. After the discussion of loglinear analysis in R × C × L contingency tables, there is a discussion of the Iterative Proportional Fitting method for estimating cell expectancies in R × C × L contingency tables under the null hypothesis of no second order interaction.

The use of loglinear modeling in the analysis of 2 × 2 contingency tables resulted from investigations of

cross-product ratios which are functions of the four cells (Bishop, Fienberg, & Holland, 1975). Consider two underlying variables A and B for a 2 × 2 contingency table and label the four cell probabilities in the following manner:

		B	
		1	2
A	1	p_{11}	p_{12}
	2	p_{21}	p_{22}

Let $p_{i+}= \sum_j p_{ij}$, $p_{+j}= \sum_i p_{ij}$, and assume $p_{++}= \sum\sum p_{ij}=1$.

A variety of functions can be defined on the cell probabilities. In particular, define

$$\alpha=(p_{11}p_{22})/(p_{12}p_{21}) \tag{56}$$

When A and B are independent, then

$$p_{ij}= p_{i+}p_{+j}, \tag{57}$$

for all i,j. When A and B are dependent, then Equation 57 is not true for some i and j. Using Equation 57, it can be shown that when A and B are independent, then Equation 56 reduces to: $\alpha=1$. In addition, when rows and columns are interchanged or when rows and columns are multiplied by a nonzero constant, the value of α does not change (i.e. α is invariant). Therefore, the following α-functions can be defined

$$\alpha_1=(p_{11}p_{12})/(p_{22}p_{21}) \tag{58}$$

$$\alpha_2=(p_{11}p_{21})/(p_{22}p_{12}) \tag{59}$$

$$\alpha_3=(p_{11}p_{22})/(p_{12}p_{21}) \tag{60}$$

By use of two basic properties of logarithms (i.e. $Ln(MN)=Ln(M)+Ln(N)$, $Ln(M/N)=Ln(M)-Ln(N)$), Equations 58 through 60 can be written

$$Ln(\alpha_1)=Ln(p_{11})+Ln(p_{12})-Ln(p_{21})-Ln(p_{22}) \quad (61)$$

$$Ln(\alpha_2)=Ln(p_{11})+Ln(p_{21})-Ln(p_{22})-Ln(p_{12}) \quad (62)$$

$$Ln(\alpha_3)=Ln(p_{11})+Ln(p_{22})-Ln(p_{12})-Ln(p_{21}) \quad (63)$$

When A and B are independent, then $\alpha_1=\alpha_2=\alpha_3=1$ and $Ln(\alpha_1)=Ln(\alpha_2)=Ln(\alpha_3)=0$. Therefore,

$$0=Ln(p_{11})+Ln(p_{12})-Ln(p_{21})-Ln(p_{22}) \quad (64)$$

$$0=Ln(p_{11})-Ln(p_{12})+Ln(p_{21})-Ln(p_{22}) \quad (65)$$

$$0=Ln(p_{11})-Ln(p_{12})-Ln(p_{21})+Ln(p_{22}) \quad (66)$$

Equations 64 through 66 can be regarded as linear contrasts that would occur in applications of analysis of variance. Since it is assumed that $p_{++}=1$, Equations 64 through 66 and $p_{++}=1$ completely determine the four cell probabilities. This suggests that a linear model in the natural logarithms of the cell probabilities be used to analyze 2 × 2 tables. The general model is defined

$$Ln(p_{ij})= U + U_{1(i)} + U_{2(j)} + U_{12(ij)} \quad (67)$$

where U (the grand mean of the logarithms of the cell probabilities) is

$$U=\{\Sigma\Sigma\ Ln(p_{ij})\}/4, \quad (68)$$

where $U + U_{1(i)}$ (the mean of the logarithms of the cell probabilities at level i of A) is

$$U + U_{1(i)}= \{Ln(p_{i1})+Ln(p_{i2})\}/2, \quad (69)$$

and where $U + U_{2(j)}$ (the mean of the logarithms of the

cell probabilities at level j of B) is

$$U + U_{2(j)} = \{Ln(p_{1j}) + Ln(p_{2j})\}/2 \quad (70)$$

The U-terms (called parameters) represent the effects of the variables and are analogous to the effects that would arise in a 2 × 2 factorial design in analysis of variance.

Since $U_{1(i)}$ and $U_{2(j)}$ represent deviations from the grand mean, it follows

$$\sum_i U_{1(i)} = \sum_j U_{2(j)} = 0 \quad (71)$$

Similarly, $U_{12(ij)}$ represents a deviation from $U + U_{1(i)} + U_{2(j)}$ and therefore

$$\sum_i U_{12(ij)} = \sum_j U_{12(ij)} = 0 \quad (72)$$

In the usual analysis of variance terminology, $U_{12(ij)}$ is an interaction term.

Using Equations 61 through 63 and Equations 68 through 70, relationships can be set up between the α-functions and the U-parameters.

From Equation 69

$$U_{1(1)} = \{Ln(p_{11}) + Ln(p_{12})\}/2 - U \quad (73)$$

Substituting Equation 68 into Equation 73

$$U_{1(1)} = (2Ln(p_{11}) + 2Ln(p_{12}) - \{Ln(p_{11}) + Ln(p_{12}) + Ln(p_{21}) + Ln(p_{22})\})/4 \quad (74)$$

$$U_{1(1)} = \{Ln(p_{11}) + Ln(p_{12}) - Ln(p_{21}) - Ln(p_{22})\}/4 \quad (75)$$

From Equation 61

$$U_{1(1)} = \{Ln(\alpha_1)\}/4 \quad (76)$$

Using similar arguments, the following relationships can be formed

$$U_{1(2)} = \{-Ln(\alpha_1)\}/4 \tag{77}$$

$$U_{2(1)} = \{Ln(\alpha_2)\}/4 \tag{78}$$

$$U_{2(2)} = \{-Ln(\alpha_2)\}/4 \tag{79}$$

Now consider the relationships between the $U_{12(ij)}$ parameters and the α-functions.

From Equation 67

$$U_{12(11)} = Ln(p_{11}) - \{U + U_{1(1)} + U_{2(1)}\} \tag{80}$$

Using Equations 69 and 70

$$\begin{aligned} U_{12(11)} = Ln(p_{11}) &- (\{Ln(p_{11})+Ln(p_{12})\}/2 \\ &+ \{Ln(p_{11})+Ln(p_{21})\}/2 \\ &- \{Ln(p_{11})+Ln(p_{12})+Ln(p_{21})+Ln(p_{22})\}/4) \end{aligned} \tag{81}$$

$$\begin{aligned} U_{12(11)} = Ln(p_{11}) &- \{2Ln(p_{11})+2Ln(p_{12})+2Ln(p_{11})+2Ln(p_{21}) \\ &-Ln(p_{11})-Ln(p_{12})-Ln(p_{21})-Ln(p_{22})\}/4 \end{aligned} \tag{82}$$

$$U_{12(11)} = \{Ln(p_{11})+Ln(p_{22})-Ln(p_{12})-Ln(p_{21})\}/4 \tag{83}$$

From Equation 63

$$U_{12(11)} = \{Ln(\alpha_3)\}/4 \tag{84}$$

Similarly

$$U_{12(12)} = -U_{12(11)} = \{-Ln(\alpha_3)\}/4 \tag{85}$$

$$U_{12(21)} = -U_{12(11)} = \{-Ln(\alpha_3)\}/4 \tag{86}$$

$$U_{12(22)} = U_{12(11)} = \{Ln(\alpha_3)\}/4 \tag{87}$$

By use of Equations 76 through 79 and Equations 84 through 87, the model specified by Equation 67 can be

written

$$Ln(p_{11}) = U + \{Ln(\alpha_1)\}/4 + \{Ln(\alpha_2)\}/4 + \{Ln(\alpha_3)\}/4 \quad (88)$$

$$Ln(p_{12}) = U + \{Ln(\alpha_1)\}/4 - \{Ln(\alpha_2)\}/4 - \{Ln(\alpha_3)\}/4 \quad (89)$$

$$Ln(p_{21}) = U - \{Ln(\alpha_1)\}/4 + \{Ln(\alpha_2)\}/4 - \{Ln(\alpha_3)\}/4 \quad (90)$$

$$Ln(p_{22}) = U - \{Ln(\alpha_1)\}/4 - \{Ln(\alpha_2)\}/4 + \{Ln(\alpha_3)\}/4 \quad (91)$$

Note that if Equations 88 through 91 were summed up, the effects would drop out, which is a characteristic of linear modeling in the usual analysis of variance design.

Since $p_{ij} = m_{ij}/n$ (m_{ij} is the expected value of cell i,j) and since the αs are invariant under row and column multiplication, each p_{ij} can be replaced by its respective m_{ij}/n and the left-hand sides of Equations 88 through 91 can be written: $Ln(m_{ij}) - Ln(n)$. If the Ln(n)s are brought to the right-hand sides of Equations 88 through 91, and if $U' = U + Ln(n)$, and if U' is changed back to U, then the general form of Equations 88 through 91 becomes

$$Ln(m_{ij}) = U + U_{1(i)} + U_{2(j)} + U_{12(ij)} \quad (92)$$

In an analysis of variance experiment a linear model is hypothesized, a test of significance is performed, and the null hypothesis is either rejected or not rejected. Similarly, in loglinear analysis, a linear model in the form of Equation 92 is hypothesized, expected values are found, a goodness-of-fit test is performed, and the model is rejected or not rejected.

For example, suppose the null hypothesis is that the

variables are independent. In terms of Equation 92, this is equivalent to the null hypothesis, H_o: $U_{12}= 0$. The expected values can be found by $m_{ij}=(n_{i+}n_{+j})/n$ and either Equation 1 or Equation 2 can be used as a test for goodness-of-fit on (2-1)(2-1) degrees of freedom. If the model is not rejected, additional tests of significance can be performed on models formed by eliminating one or more parameters in Equation 92.

The extension of loglinear analysis to the R × C contingency table is a simple matter of letting i and j range from 1 to R and from 1 to C, respectively.

For a 2 × 2 × 2 contingency table with $p_{+++}=1$, there is an additional variable and additional parameters. Equation 92 becomes

$$\begin{aligned} Ln(m_{ijk}) &= U + U_{1(i)} + U_{2(j)} + U_{3(k)} \\ &\quad + U_{12(ij)} + U_{13(ik)} + U_{23(jk)} + U_{123(ijk)} \end{aligned} \quad (93)$$

In Equation 93, $U_{3(k)}$ is the main effect of the third independent variable, $U_{23(jk)}$ and $U_{13(ik)}$ are the interaction effects between variables 2 and 3 and between variables 1 and 3, and $U_{123(ijk)}$ is the second order interaction effect. As in analysis of variance

$$\sum_i U_{1(i)} = \sum_j U_{2(j)} = \sum_k U_{3(k)} = 0,$$

$$\sum_i U_{12(ij)} = \sum_j U_{12(ij)} = \sum_i U_{13(ik)}$$

$$= \sum_k U_{13(ik)} = \sum_j U_{23(jk)} = \sum_k U_{23(jk)} = 0,$$

$$\sum_i U_{123(ijk)} = \sum_j U_{123(ijk)} = \sum_k U_{123(ijk)} = 0 \qquad (94)$$

As in the 2 × 2 contingency table, the extension of 2 × 2 × 2 to the R × C × L contingency table is a matter of letting the subscripts range over all possible values (i.e. i=1,2,...,R; j=1,2,...,C; k=1,2,...,L). Therefore Equation 93 can represent just as well the loglinear model for the R × C × L contingency table as it represents the loglinear model for the 2 × 2 × 2 contingency table.

A null hypothesis for an R × C × L contingency table can be tested in a manner analogous to the procedure used for testing a null hypothesis in an R × C contingency table. A model based upon the null hypothesis is proposed, the expected values are found, Equation 1 or Equation 2 can be used to test for a goodness-of-fit, and H_o is or is not rejected. If H_o is not rejected, then alternative models can be hypothesized and tests of goodness-of-fit can be carried out again. For example, if the null hypothesis is "no second order interaction" (i.e. $U_{123}=0$), then Equation 93 is the loglinear model; but without the U_{123} parameter. H_o will be rejected or not rejected as a result of the goodness-of-fit test on (R-1)(C-1)(L-1) degrees of freedom (Fienberg, 1977).

A loglinear model that has all possible parameters is called a saturated model (Bishop, Fienberg, & Holland, 1975) and the maximum number of parameters is equal to the

number of cells. For example, Equation 92 is the saturated model for the 2 × 2 contingency table and Equation 93 is the saturated model for the 2 × 2 × 2 contingency table. On the other hand, an unsaturated model is one in which one or more parameters are missing. For example, the model which corresponds to the null hypothesis of no second order interaction (i.e. $U_{123}=0$) is

$$Ln(m) = U + U_1 + U_2 + U_3 + U_{12} + U_{13} + U_{23} \quad (95)$$

where the i,j,k subscripts have been dropped since it is understood that i,j,k range over their respective R, C, and L values.

Loglinear models also can be classified as being hierarchical or nonhierarchical. This concept is derived from the relationships among the numerical subscripts (see Equation 95). Suppose one U-term has r subscripts and another U-term has s subscripts with r>s. These terms are called relatives if the r subscripts contain among them all the s subscripts (Bishop, Fienberg, & Holland, 1975). For example, U_{123} is a higher order relative of all other U-terms and U_{12} is a higher order relative of U_1 and U_2, but U_{12} is not a relative of U_3. Therefore, a term such as U_{123} cannot be included in a model unless U_{12}, U_{13}, and U_{23} are in the model.

If a model is nonhierarchical then a transformation can be used to change the model into a hierarchical model (Fienberg, 1977). For the most part, therefore, loglinear

analysis of contingency tables is done with respect to hierarchical models.

The final topic of discussion in loglinear analysis is the procedure for estimating cell frequencies. The maximum likelihood estimate (MLE) method is used to compute cell estimates. There are three practical advantages in using MLEs. First, for loglinear models, MLEs are relatively easy to compute. Second, MLEs satisfy intuitive marginal constraints. Third, MLEs are applicable to multinomial data when some observed frequencies are zeros. This third point is useful when a sample is small (Bishop, Fienberg, & Holland, 1975).

Let the MLE of n_{ijk} be denoted by $\hat{m}_{ijk}$ and let n_{ij+}, n_{i+k}, and n_{+jk} have meanings which are analogous to p_{ij+}. Let $\{n_{ij+}\}$, $\{n_{i+k}\}$, and $\{n_{+jk}\}$ be the respective R × C, R × L, and C × L rectangular tables and denote these tables by C_{12}, C_{13}, and C_{23}, respectively. C_{ij} is called a configuration.

For the model which hypothesizes no second order interaction in an R × C × L contingency table, C_{12}, C_{13}, and C_{23} are sufficient configurations. This means C_{12}, C_{13}, and C_{23} are the minimal statistics for obtaining the MLEs of n_{ij+}, n_{i+k}, and n_{+jk}, respectively (Birch, 1963).

Furthermore, Birch (1963) proved that when sampling is done from a multinomial distribution, there is a unique set of m_{ijk}s that satisfy both the marginal constraints

given by the C_{ij}s and the null hypothesis of no second order interaction.

The C_{ij}s are important because they are used to determine whether cell estimates can be found directly or whether cell estimates must be approximated by means of an iterative procedure. For the null hypothesis of no second order interaction in an R × C × L contingency table, C_{12}, C_{13}, and C_{23} form a closed loop. This means that any two configurations can be connected by overlapping subscripts to give the third configuration. For example, C_{23} and C_{12} overlap at "2" and therefore, these lead to C_{13}. When closed loops exist, then an iterative method must be used to find cell estimates (Bishop, Fienberg, & Holland, 1975).

MLEs of cells of hierarchical models can be obtained by using the Iterative Proportional Fitting method on the set of minimal configurations. There are five advantages in using the Iterative Proportional Fitting method. These are:

1) It always converges to the required MLEs of the n_{ijk}s.

2) Cell estimates can be found to any desired degree of accuracy, δ (in the present study $\delta=1.0\times10^{-6}$).

3) Estimates depend on only the minimal set of configurations and no provision need be made for some cells that have observed frequencies of zero.

4) Any set of starting values may be choosen that conform to the model being fitted.

5) If direct estimates exist, then the method will find them in one cycle (Bishop, Fienberg, & Holland, 1975).

The Iterative Proportional Fitting method was used in 1940 to analyze census data (Deming & Stephan, 1940). The procedure uses a least squares adjustment of the observed frequencies while keeping the marginal totals fixed.

For example, suppose the null hypothesis of an $R \times C \times L$ contingency table is H_o: $U_{123}=0$. The procedure begins by choosing initial values of the $\hat{m}_{ijk}$s that satisfy the null hypothesis of no second order interaction. A convenient choice of initial values are $\hat{m}_{ijk}^{(0)}=1$ for all i,j,k where the exponent represents the initial step. These preliminary estimates are adjusted to fit C_{12}, C_{13}, and C_{23} in the following manner.

$$C_{12}: \quad \hat{m}_{ijk}^{(1)}=(\hat{m}_{ijk}^{(0)}n_{ij+})/\hat{m}_{ij+}^{(0)} \tag{96}$$

$$C_{13}: \quad \hat{m}_{ijk}^{(2)}=(\hat{m}_{ijk}^{(1)}n_{i+k})/\hat{m}_{i+k}^{(1)} \tag{97}$$

$$C_{23}: \quad \hat{m}_{ijk}^{(3)}=(\hat{m}_{ijk}^{(2)}n_{+jk})/\hat{m}_{+jk}^{(2)} \tag{98}$$

This completes the first 3-step cycle. The process is repeated until a complete cycle does not cause any cell to change by a pre-assigned value, δ.

Since 1940, a number of authors used this algorithm (i.e. Iterative Proportional Fitting) and gave proofs of the convergence of the solution to unique cell estimates

(Ku & Kullback, 1968; Shaffer, 1971; Darroch & Ratcliff, 1972).

Summary of Methods for Testing of Second Order Interaction

This concludes the section which discusses the three methods (Bartlett, Goodman, Iterative Proportional Fitting) used in the present study to test the null hypothesis of no second order interaction in an R × C × L contingency table. The methods differ in the manner in which H_o is tested. The Bartlett method obtains cell estimates by solving systems of equations. The Goodman method does not obtain cell estimates, but rather tests H_o by means of multivariate normal theory. The Iterative Proportional Fitting method is used in conjunction with loglinear modeling to obtain cell estimates by making proportional adjustments without the necessity of solving systems of equations.

Rationale for Levels of Sample Size

In the present study, the levels of the independent variable sample size were 20, 40, 60, 80, 100. The justification for using these levels is based upon two considerations. First, two recent studies (Larntz, 1978; Cox & Plackett, 1980) used the sample sizes 20, 40, 60, 80, and 100 in investigations of the behavior of various statistics that are used to approximate χ^2 (e.g. X^2, L^2, see Equations 1 and 2). Second, the amount of computer

time that was available had a direct bearing on the number of levels of sample size. It was necessary to restrict the number of sample sizes to a number which would be manageable with respect to computer time but which still would produce a maximum factor of generalization for the present study.

Rationale for Levels of Dimension

The dimensions of the contingency tables used in the present study (2 x 2 x 2, 2 x 2 x 3, 3 x 3 x 3) were determined by the literature related to small sample studies of R x C x L contingency tables. In particular, the studies of Odoroff (1970), McNamee (1973), Korducki (1977), Larntz (1978), and Cox and Plackett (1980) served as guidelines for choosing the three levels of dimension. Odoroff's (1970) comparative study of 12 goodness-of-fit statistics was conducted on 2 x 2 x 2 and 2 x 2 x 3 contingency tables. McNamee's (1973) investigation of second order interaction was done with 2 x 2 x L contingency tables where L ranged from 2 to 12. Korducki's (1977) limited Monte Carlo study of the behavior of X^2 and L^2 (see Equations 1 and 2) for small samples was done with 3 x 3 x 3 and 4 x 4 x 4 contingency tables. In one part of the Cox and Plackett (1980) study of X^2 and L^2 (see Equations 1 and 2), small samples were generated for 2 x 2 x L and 3 x 3 x 3 contingency tables where L was 3, 4, and 18. Therefore, the dimensions used in the

present study were 2 × 2 × 2, 2 × 2 × 3, and 3 × 3 × 3 because these three dimensions seemed to be used quite often in previous studies.

Rationale for Levels of Sampling Distribution

Underlying multinomial distributions were used to generate the contingency tables used in the present study. The multinomial distributions were chosen so that they were void of second order interaction. Therefore, a randomly generated contingency table based upon one of these distributions should indicate second order interaction by chance alone. In order to make the result more general, a variety of multinomial distributions were used. The multinomial sampling distribution used in the present study range from being highly skewed (e.g. 2:3:5, 2:3:5, 2:3:5 for rows, columns, and layers, respectively) to being nearly uniform (e.g. 6:6:7, 6:6:7, 6:6:7 for rows, columns, and layers, respectively). The multinomial sampling distributions used for the 3 × 3 × 3 contingency tables were identical to those used by Larntz (1978). The multinomial sampling distributions used for the 2 × 2 × 2 and 2 × 2 × 3 contingency tables were chosen arbitrarily; but in such a way that they ranged from being highly skewed (e.g. 1:2, 1:2, 1:2 for rows, columns, and layers, respectively) to being nearly uniform (e.g. 4:5, 4:5, 4:5 for rows, columns, and layers, respectively) and such that they exhibited no second order interaction.

Table 1 contains the sampling distributions that were used as levels of the independent variable sampling distribution in the present study. For each dimension ($2 \times 2 \times 2$, $2 \times 2 \times 3$, $3 \times 3 \times 3$) there are four multinomial sampling distributions (e.g. P-1 to P-4). Each multinomial sampling distribution gives the proportions among row levels, column levels, and layer levels. For example, in the $3 \times 3 \times 3$ contingency table, P-1 indicates that the rows are in the proportion of 2:3:5, the columns are in the proportion of 2:3:5 and the layers are in the proportion of 2:3:5. This means that the underlying theoretical first order interaction between rows and columns is the same across layers, the underlying theoretical first order interaction between rows and layers is the same across columns, and the underlying theoretical first order interaction between columns and layers is the same across rows. Theoretically, there is no second order interaction in the table.

Rationale for the Number of Iterations

The reliability of the results obtained from a Monte Carlo study is related to the number of samples (iterations), N, that were generated and N is a function of the precision required of the results and the level of significance. The number of iterations for the present study was determined by a procedure used by Kavanagh (1972) and McNamee (1973). The following is an

Table 1

Multinomial Sampling Distributions by Dimension

Dimension	Proportion	Rows	Columns	Layers
3 × 3 × 3				
	P-1	2:3:5	2:3:5	2:3:5
	P-2	6:6:7	6:6:7	6:6:7
	P-3	2:3:5	6:6:7	6:6:7
	P-4	2:3:5	2:3:5	6:6:7
2 × 2 × 3				
	Q-1	1:2	1:2	2:3:5
	Q-2	1:2	1:2	6:6:7
	Q-3	4:5	4:5	2:3:5
	Q-4	4:5	4:5	6:6:7
2 × 2 × 2				
	R-1	1:2	1:2	1:2
	R-2	1:2	1:2	4:5
	R-3	1:2	4:5	4:5
	R-4	4:5	4:5	4:5

explanation of this procedure.

Let X^2 represent the statistic used to test the null hypothesis of no second order interaction in an $R \times C \times L$ contingency table. X^2 can be either the Pearson statistic (see Equation 1) which was used in the Iterative Proportional Fitting method or the statistic used by Kastenbaum and Lamphiear (1959) (see Equation 38) in the Bartlett method. The rejection rate (the dependent variable) was analyzed with respect to the upper 5% of the χ^2 distribution. Let $\chi^2_{.95}$ denote the 95th percentile of the χ^2 distribution.

Let

$$y = \begin{cases} 1, \text{ when } X \geq \chi^2_{.95} \\ 0, \text{ when } X < \chi^2_{.95} \end{cases}$$

$\sum_{i}^{N} Y_i$ is a binomial distribution with a mean of p and a variance of $p(1-p)/N$. Let $\hat{p}$ be an estimate of p.

By the Central Limit Theorem (Hays, 1973), for large N, $\hat{p}$ is approximately normally distributed with a mean of p and a variance of $p(1-p)/N$. The following probability statement can be made for some level of significance, α.

$$\text{Prob}((\hat{p}-p)/\sqrt{p(1-p)/N} \geq z_{1-\alpha/2}) = 1-\alpha$$

For a 95% confidence interval (i.e. $\alpha = .05$), the appropriate z-value (z=1.96) was found in a standard normal table.

The precision of the present study was set at .01. This means that estimates of p were found so that

$|\hat{p}-p| \leq .01$ 95% of the time. For this precision, the following equation comes from Equation 99

$$N=1.96^2p(1-p)/.01^2 \qquad (100)$$

In order to obtain a value for N, a value for p must be substituted into Equation 100. From elementary calculus, the worst situation occurs when p=.5. In that case, N=9,604. However, if there is reason to believe that p is a value other than .5, this value may be used in Equation 100 to obtain a smaller value of N. For example, if p=.10, then N=3,457. In the present study, since p was analyzed with respect to the upper 5% of the χ^2 distribution, it was reasonable to let p=.05. For p=.05, then N=1,825. Therefore, the Monte Carlo procedure randomly generated 1,825 contingency tables for every combination of sample size (20, 40, 60, 80, 100), dimension (2 × 2 × 2, 2 × 2 × 3, 3 × 3 × 3), and sampling distribution (four per each dimension).

Procedure for Randomly Generating a Table

For a given sample size, dimension, and sampling distribution, the algorithm for filling the cells of the contingency table is as follows.

Suppose N=60, the dimension is 3 × 3 × 3, and the sampling distribution is p-1 (see Table 1). The McGill University Random Number Generator Package "Super-Duper" (Marsaglia, Anathanarayanan, & Paul, 1972) was used to randomly generate a real number between 0 and 1.

Therefore, the open interval (0,1) must be partitioned into 27 subintervals according to the given sampling distribution (i.e. P-1). Each subinterval corresponds to a cell of the 3 × 3 × 3 contingency table. First, (0,1) is partitioned according to the layer proportions (i.e. 2:3:5). Second, each of these three intervals is partitioned according to the column proportions (i.e. 2:3:5). Third, each of these nine intervals is partitioned according to the row proportions (i.e. 2:3:5). Each of the 27 intervals corresponds to one array element which contains the frequency count for the number of times a random number fell into the corresponding interval.

After the intervals have been formed, a random number is generated, the number of the interval into which it falls is noted, and the corresponding array element has its frequency count increased by one. This is repeated until 60 random numbers have been distributed among the intervals. Then the computer matches the contents of the 27 array elements with the appropriate cells of the 3 × 3 × 3 contingency table. This procedure results in a randomly generated 3 × 3 × 3 contingency table of size 60.

Procedure for Random Zeros

There are two types of observed zeros in contingency tables: Fixed zeros and random (sampling) zeros (Fienberg, 1977). A cell can have a fixed zero when it is impossible

to have an observation for a combination of the classification variables. For example, in a study of Catholic clergy in 1940, the cross-classification of female-priest has a fixed zero. A random zero can occur because of sampling variations or because of the small sample size relative to the large number of cells. In this case (random zero), if the sample size is increased sufficiently, then the random zeros will be eliminated.

Since the behavior of the three methods of testing for second order interaction (Bartlett, Goodman, Iterative Proportional Fitting) with small samples was the topic of interest in the present study, fixed zeros were not of concern. Therefore, this section of the Method chapter contains a discussion of the procedures used for handling random zeros in the Bartlett, Goodman, and Iterative Proportional Fitting methods.

Random Zeros in the Bartlett Method

In the derivation of the Bartlett algorithm for computing cell estimates, it was assumed that $x_{ijk}/n_{ijk}<1$ (see Equation 36), where x_{ijk} is the deviation and n_{ijk} is the observed value of the i,j,k cell. If n_{ijk} is zero, then the inequality ($x_{ijk}/n_{ijk}<1$) is not satisfied and the series given by Equation 36 does not converge. The recommended procedure (Norton, 1945; Kastenbaum & Lamphiear, 1959) to be used when some n_{ijk}s are zero is to initialize first estimates of the x_{ijk}s for those cells

that have $n_{ijk}=0$. Therefore, the Bartlett procedure will begin with the second iterative step. First estimates of the x_{ijk}s which have corresponding n_{ijk}s equal to zero are added to these corresponding n_{ijk}s for k=1,2,...,L. At the same time, every time a first estimate is added to a n_{ijk} (k=1,2,...,L), then a first estimate is subtracted from the largest n_{ijk} (k=1,2,...,L). In this way the condition of Equation 23 is maintained.

For the 2 × 2 × 2 contingency table, if, for example, $n_{ij1}=0$, then the initial estimate of x_{ij1} is added to n_{ij1} and subtracted from n_{ij2}. The reverse would be done if n_{ij2} were zero. For the 2 × 2 × 3 (or 3 × 3 × 3) contingency table, if, for example, $n_{ij2}=0$, then the initial estimate of x_{ij2} is added to n_{ij2} and subtracted from the larger of n_{ij1} or n_{ij3}. If $n_{ij1}= n_{ij3}$, then either one may be designated the larger value. If two observed values are zero, for example n_{ij1} and n_{ij3}, then let the initial esimate of x_{ij1} equal the initial estimate of x_{ij3}. Add this initial estimate to both n_{ij1} and n_{ij3} and subtract this initial estimate twice from n_{ij2}.

For the 2 × 2 × 2 contingency table, the initial estimate was .5. For the 2 × 2 × 3 (or 3 × 3 × 3) table, the initial estimate was .75. The initial estimates differed for the following reason. For L=3, if two observed values are zero, then a first estimate of .5 would result in 1.0 being subtracted from the third

nonzero observed cell. Now, if this third nonzero observed cell had a value of 1, then the adjustment would result in a zero in that cell and the series in Equation 36 would not converge.

It may happen that $n_{ijk}=0$ for all $k=1,2,\ldots,L$. In this case the adjustment procedure fails and the series given by Equation 36 will not converge. The only other alternative (other than to exclude the table in the calculation of the percentage of rejection) is to make use of the symmetric property of the Bartlett method (Simpson, 1951). Columns were interchanged within each layer for the 2 × 2 × 2 and 2 × 2 × 3 contingency tables and rows and columns were interchanged within each layer for the 3 × 3 × 3 contingency table (McNamee, 1973). For the 2 × 2 × 2 and 2 × 2 × 3 contingency tables there is only one possible interchange of columns and for the 3 × 3 × 3 contingency table there are eight possible interchanges of either rows or columns. If both the adjustment procedure and the interchange procedure still resulted in the table not being analyzed, then that table was not used in the calculation of the number of times the Bartlett method rejected for that particular set of 1,825 tables.

Random Zeros in the Goodman Method

In the Goodman method, in order to proceed from Equation 41 to Equation 42, the natural logarithm of a

quotient is taken. If the quotient is zero or if the denominator of the quotient is zero, then the natural logarithm is undefined. Since the Goodman method is not based upon a Bartlett-type iterative procedure, there is no provision for an adjustment to cells with zeros. There are two options available. One option is to reject the table when calculating the proportion of times the Goodman method rejected a given set of 1,825 tables. The second option (the one used in the present study) is to eliminate the zeros by adding a constant to those cells that have zeros for their observed values. The choice of constant to be added and the question of whether to add the constant to just those cells having zeros or to all cells have not been resolved (Reynolds, 1977). Therefore, in the present study the Goodman method was applied to each table twice. For the first analysis, ½ was added to only those cells that had observed values of zero. For the second analysis, ½ was added to every observed value (Goodman, 1964a; Reynolds, 1977).

Random Zeros in the Iterative Proportional Fitting Method

A random zero by itself does not enter into the three cycle procedure for estimating cell frequencies by the Iterative Proportional Fitting method (see Equation 96, 97, 98). However, random zeros are important factors in this method when they lead to marginal totals of zero

in the given table of observed values. These marginal totals are the entries in the configurations. If at least one marginal total is zero, then at least one expected marginal total will be zero. Therefore, in the next step of the three-cycle procedure, there will be at least one division by zero.

Fienberg (1977) suggested the following procedure when some observed marginal totals are zero. First, define zero divided by zero as zero. Therefore, if an observed marginal total is zero, then all entries that add up to that total will remain zero during the iterative procedure. Second, the degrees of freedom must be adjusted because if an observed marginal entry is zero, then both the expected and the observed cell entries for all cells of that marginal total must be zero. Therefore, the fit of those observed cells is perfect and the degrees of freedom associated with the fit of zero cell values must be deleted.

Fienberg (1977) gave the following general formula for computing the adjusted degrees of freedom:

$$df = (T_e - Z_e) - (T_p - Z_p), \qquad (101)$$

where:

T_e= # cells in the table that are being fitted

T_p= # parameters fitted by the model

Z_e= # cells which contain zero estimated expected values

Z_p= # parameters that cannot be estimated because of marginal totals equal to zero

For example, consider the following 2 × 2 × 2 contingency table.

K=1		K=2	
n_{111}=0	n_{121}=5	n_{112}=6	n_{122}=9
n_{211}=0	n_{221}=6	n_{212}=5	n_{222}=7

Here, n_{+11}=0. If n_{+11} is used to get estimates for a given model, then knowledge of n_{111} is sufficient to determine n_{211}, and conversely. Therefore there is one degree of freedom associated with the cells that add up to n_{+11}, and it must be subtracted from the total degrees of freedom whenever n_{+11} is zero. If the model fitted is H_0: U_{123}=0, then there is (2-1)(2-1)(2-1)=1 degree of freedom. But since n_{+11}=0, then the adjusted degrees of freedom is 0. Fienberg (1977) said that the fit of the model is perfect whenever the adjusted degrees of freedom is zero. For this example, T_e=8, T_p=7, Z_e=2, and Z_p=1. Therefore the degree of freedom is 0 (see Equation 101).

According to Bishop, Fienberg, and Holland (1975), it is possible for Equation 101 to result in negative degrees of freedom. However, Haberman (1980) said that logically this cannot happen. When negative degrees of freedom do occur, this indicates that the model is inappropriate for the data. In other words, the model is not linear. Equation 101 was incorporated in the computer program that

was written for the methods of testing for second order interaction (Bartlett, Goodman, Iterative Proportional Fitting).

Summary of Procedures Used for Random Zeros

This concludes the discussion of the procedures used when random zeros occurred in the contingency tables. The Goodman method is the only method that requires a constant to be added to those cells which have frequencies of zero. Although the addition of a constant may not be that important in large samples, the consequences of adding a constant to small samples are unknown. The Iterative Proportional Fitting method seems to be the most flexible when some cells have frequencies of zero. However, the method is open to criticism when "zero divided by zero" is defined as zero. The Bartlett method is the most conservative when it comes to making adjustments for zero frequencies. However, the adjustments are made not to the cells, but to the cell deviations. Consequently, the original observed frequencies are not altered.

Procedures for Negative Chi-Square

Theoretically, it is impossible for the test statistics used in the present study (see Equations 1, 38, 46) to be negative. However in the Bartlett method, the series given in Equation 36 might not converge and therefore the statistic given by Equation 38 could be negative (McNamee, 1973). Whenever this occurred, the "interchange procedure" described

in the Random Zeros in the Bartlett Method section of the Method chapter was used. If negative values persisted, then those tables were eliminated when the rejection rate for that set of 1,825 contingency tables was computed.

In the Goodman method, negative values of the statistic given by Equation 46 can occur because of rounding error due to the computer. If $Y^2 > -.01$ (see Equation 46), then the negative value was attributed to rounding error and the table was included in the calculation of the rejection rate. Since the Goodman method is not invariant under row and column interchanges (Goodman, 1964b), if $Y^2 \leq -.01$, then the table was dropped from the calculation of the rejection rate.

In the Iterative Proportional Fitting method, the three-cycle procedure (see Equations 96 to 98) for computing MLEs of cell expectancies involved adjustments based upon marginal totals. The marginal totals are always nonnegative numbers and thus the expected values obtained by this procedure must be nonnegative numbers and it is impossible for X^2 (see Equation 1) to be negative.

Procedures for Computer-Related Problems

Computer related problems fell into two categories. The first category consisted of technical problems peculiar to the method (Bartlett, Goodman, Iterative Proportional Fitting) used to test for second order interaction.

In the Bartlett method, the most important problem

was the convergence of the series in Equation 36. The key to convergence is the behavior of the adjustments found by Equation 33. If these adjustments increase in absolute value, then the corresponding expectations will increase in absolute value. Since the maximum sample size was 100 and the minimum number of cells was eight (2 × 2 × 2), then the average observed frequency per cell would be about 12. For skewed distributions, some observed frequencies would be higher while other observed frequencies would be lower. But in no case can each cell of any table have expectations in excess of 50. Therefore, if cell expectations were greater than 50 (in absolute value), then this would indicate that the series in Equation 36 did not converge. When the series did not converge, the "interchange procedure" described in the Random Zeros in the Bartlett Method section of the Method chapter was used.

The Goodman method relies on computing the inverses of a number of matrices (see Equations 47 or 48). In this case, two errors can be made. First, the computer may be in error by indicating that the given matrix is singular (i.e. no inverse exist) when in fact the inverse does exist. Second, the computer may be in error by finding the inverse of a given matrix when in fact the matrix is singular. Both errors occur because of the accuracy of the computer. Therefore it is important that a high quality computer subroutine be used to calculate the inverses. The matrix

inversion procedure suggested by Forsythe and Moler (1967) was used in the present study. This algorithm is used by the Satistical Analysis System Institute in its matrix procedure section (SAS User's Guide, 1979).

The second category of problems consisted of those technical problems that were of a general nature regardless of the particular method (Bartlett, Goodman, Iterative Proportional Fitting) used to test for second order interaction. There were two types of technical problems in this category.

The first technical problem was the "divide-check" problem. If the divisor of a quotient is truly zero, then the computer used in the present study (IBM 360/370) does not attempt the division. Instead, the computer issues a warning of the upcoming division by zero, sets the result of what would have been a division by zero to zero, and continues executing the program. Consequently, any results based upon this artificial zero will not be true. To solve this problem, a "divide-check" subroutine was inserted in the computer program wherever a true division by zero could occur. This subroutine caused the computer to stop evaluating the given table and either go to the "interchange procedure" if the problem occurred in the Bartlett method or droppped the table from the calculation of the rejection rate if the problem occurred in the Goodman method or Iterative Proportional Fitting method.

The second type of technical problem was the "overflow-

underflow" problem. This problem occurs when the computer actually performs the arithmetic operation (usually division) and the result either is too large for the machine or too small for the machine. In the IBM 360/370, the result is set to zero and the computer continues to execute the program. As in the "divide-check" problem, the results are meaningless. One way to solve this problem is to put artificial restrictions on the magnitude of the numbers. In this way the "overflow-underflow" problem would never occur. If artificial restrictions were placed on the magnitude of the numbers, then the results of the present study would be interpreted in terms of these restrictions. The alternative to placing artificial restrictions on the magnitude of the number is to use an "overflow-underflow" subroutine which is analogous to the "divide-check" subroutine. In other words, should an "overflow-underflow" problem occur, then the computer would stop evaluating the given table and attempt either the "interchange procedure" if the problem occurred in the Bartlett method or drop the table from the calculation of the rejection rate if the problem occurred in the Goodman method or Iterative Proportional Fitting method. In the present study, the "overflow-underflow" subroutine was used to solve this problem.

Design and Statistical Analysis

1,825 contingency tables were randomly generated

for every combination of sample size (20, 40, 60, 80, 100), dimension (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2), and sampling distribution (four per dimension). This resulted in 60 combinations of 1,825 contingency tables. Each table was tested for second order interaction by each method (Bartlett, Goodman, Iterative Proportional Fitting). The statistical analysis consisted of four parts: the analysis of the percentage of chi-square statistics which were beyond the theoretical 5% point of the χ^2 distribution (i.e, rejection rates), univariate analysis of the Bartlett, Goodman, and Iterative Proportional Fitting statistics, regression analysis of the Bartlett, Goodman, and Iterative Proportional Fitting statistics, and a discussion of the tables not analyzed by the methods.

Analysis of Rejection Rates

Rejection rates of the null hypothesis, no second order interaction, were computed for those combinations of sample size, dimension, and sampling distribution specified in the Hypotheses section of the Method chapter. If the methods are robust for small samples or if combinations of sample size, dimension, and sampling distribution have no affect on the rejection rates, then the statistics (see Equations 1, 38, 46) should exceed the tabled value of χ^2 distribution about 5% of the time. A method was considered to be robust when its rejection rate (e.g. Bartlett rejection rate for the 2 × 2 × 3 dimension and

sample size 40) was within .01 of this .05 value.

Univariate Analysis

Univariate descriptive statistics were found for each method (Bartlett, Goodman, Iterative Proportional Fitting) for each combination of sample size, dimension, and sampling distribution. Specifically, the means and variances of the chi-square statistics of the methods (see Equations 1, 38, 46) were computed for the 60 combinations of sample size (5), dimension (3), and sampling distribution (4). Each mean and variance was compared to the theoretical mean and variance of the corresponding χ^2 distribution for the appropriate degrees of freedom. For example, for a given sample size and sampling distribution, the theoretical distribution of the chi-square statistics of a $2 \times 2 \times 2$ contingency table should have a mean of 1 and a variance of 2 (Hays, 1973). Therefore, it may be of interest to note how well the mean and variance of each method compares to the mean and variance of the corresponding χ^2 distribution.

In addition, the data for the chi-square statistics were collapsed on sampling distribution and the means and variances were computed for the 15 combinations of sample size (5) and dimension (3). The rationale for collapsing on sampling distribution is that from a practical point of view the results of this particular study would be more meaningful in terms of sample size and

dimension than in terms of the finer distinction brought in by the inclusion of sampling distribution.

Linear Regression Analysis

For each combination of sample size (5) and dimension (3), linear regression techniques were used to search for relationships among the Bartlett, Iterative Proportional Fitting, and Goodman chi-square statistics (see Equations 1, 38, 46). The data were collapsed on sampling distribution because the results would be more meaningful in terms of sample size and dimension than in terms of the finer distinction brought in by the inclusion of sampling distribution.

The linear regression analysis consisted of two parts. In part one, the Goodman chi-square statistic (see Equation 46) was used as the predictor and the Bartlett and Iterative Proportional Fitting chi-square statistics were used as the dependent variables. Since the Goodman chi-square statistic was computed twice for each contingency table (first by adding ½ to cell frequencies that were zero, Goodman 1, and second by adding ½ to every cell frequency, Goodman 2), the regression analysis was done twice. The Goodman chi-square statistic was used as the predictor because it analyzed all contingency tables and the Bartlett and Iterative Proportional Fitting methods did not analyze all contingency tables. It was hoped that the regression analysis would lead to a

"correction factor" which could be used to predict the Bartlett or Iterative Proportional Fitting chi-square statistics when the latter methods failed to analyze a contingency table.

In part two of the regression analysis, the Iterative Proportional Fitting chi-square statistic (see Equation 1) was used as the predictor and the Bartlett chi-square statistic (see Equation 38) was used as the dependent variable. The Iterative Proportional Fitting chi-square statistic was used as the predictor because it analyzed more tables than did the Bartlett method. It was hoped that the regression analysis would lead to a "correction factor" which could be used to predict the Bartlett chi-square statistic when the latter method failed to analyze a contingency table.

Discussion of Unanalyzed Tables

The Bartlett, Goodman, and Iterative Proportional Fitting methods may not be able to test a contingency table for second order interaction. The Bartlett method cannot be used to test for second order interaction wherever the series in Equation 36 fails to converge. The Goodman method cannot be used to test for second order interaction whenever a matrix used in the procedure is nearly singular (see Goodman Method in Method chapter). The Iterative Proportional Fitting method cannot be used to test for second order interaction whenever the adjusted

degrees of freedom (see Equation 101)is less than or equal to zero. Therefore, the last part of the Results chapter contains a discussion of the tables not analyzed by the methods (i.e. Bartlett, Goodman, Iterative Proportional Fitting).

The statistical procedures found in the SAS User's Guide, 1979 Edition of the Statistical Analysis System Institute were used to compute the univariate descriptive statistics and to perform the regression analysis.

Summary of the Method Chapter

This concludes the discussion of the Method chapter. The hypotheses, the rationale for the present study, detailed explanations of the procedures used in the present study, and the procedures used to conduct the analysis of the data generated by the Monte Carlo technique were described in this chapter. The rejection rates for the Bartlett, Goodman, and Iterative Proportional Fitting methods, the univariate descriptive statistical results, the linear regression results, and the discussion of the unanalyzed tables are given in Chapter Four.

CHAPTER IV

RESULTS

This chapter contains the results of the present study. The chapter consists of four parts. Part one contains the results of the rejection rates of the null hypothesis (H_o), no second order interaction in the R × C × L contingency table. Part two contains the results of the univariate descriptive statistics that were calculated for the Bartlett, Iterative Proportional Fitting (IPF), and Goodman methods. Part three contains the results of the linear regression analysis conducted on the Bartlett, IPF, and Goodman methods. Part four contains a discussion of the tables rejected for analysis by the Bartlett, IPF, and Goodman methods.

Results of Rejection Rates

This part of the Results chapter contains the results of the rejection rates of the null hypothesis, no second order interaction in the R × C × L contingency table. The results are reported in terms of the six hypotheses presented on pages 49-50 of Chapter III, Method.

Hypothesis 1

Hypothesis 1 is: The rejection rates of the methods are the same. For this hypothesis the overall rejection rates of the methods (Bartlett, IPF, Goodman) were computed

while the independent variables (sample size, dimension, and sampling distribution) were ignored. Since there were five sample sizes (20, 40, 60, 80, 100), three dimensions (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2), and four sampling distributions per dimension (see Table 1), there were 60 {i.e. 3(5)(4)} sets of contingency tables. For each set, 1,825 contingency tables were generated randomly. This resulted in 109,500 randomly generated contingency tables {i.e. 60(1,825)}. There was an attempt made to analyze each table for second order interaction by means of the three methods. Since there was a question as to which procedure to follow in the Goodman method when a cell frequency is zero (see Random Zeros in the Goodman Method in the Method chapter), each table was analyzed twice by the Goodman method. In the first analysis (Goodman 1), a ½ was added to each cell frequency that was zero. In the second analysis (Goodman 2), a ½ was added to every cell frequency.

For each method (Bartlett, IPF, Goodman 1, Goodman 2), Table 2 contains the rejection rates and the number of tables actually analyzed by the methods. For example, of the 109,500 contingency tables that were generated randomly, the Bartlett rejection rate was .049 and the Bartlett method analyzed 69,825 contingency tables.

Hypothesis 2

Hypothesis 2 is: For a given sample size, the

Table 2

Theoretical .05 vs. Calculated Rejection Rates by Method

Method	Rej. Rate	Number Analyzed[a]
Bartlett	.049	69,825
IPF	.066	104,815
Goodman 1	.016	109,500
Goodman 2	.014	109,500

[a]Out of a possible 109,500 tables

rejection rates of the methods are the same. For this hypothesis, the 109,500 contingency tables were classified according to sample size. There were five sample sizes (20, 40, 60, 80, 100) with each level of sample size containing the same number of contingency tables. Hence, there were 21,900 contingency tables in each level of sample size.

For each sample size, Table 3 contains the rejection rate and the number of tables actually analyzed for each method. For example, of the 21,900 contingency tables and for a sample size of 20, the Bartlett rejection rate was .036 and the Bartlett method analyzed 5,774 tables. For the same sample size, the IPF rejection rate was .088 and the IPF method analyzed 18,191 tables. For a sample size of 20, the Goodman 1 rejection rate was .000 (to three decimal places) and 21,900 tables were analyzed. The Goodman 2 rejection rate was .001 and 21,900 tables were analyzed for the sample size 20.

In addition to containing the results for a given sample size, Table 3 contains the rejection rates of a given method as the sample size increases from 20 to 100. For example, reading down the Bartlett column, for a sample size of 20 the Bartlett rejection rate was .036 for 5,774 tables analyzed, for a sample size of 40 the rejection rate was .049 for 12,252 tables analyzed, etc.

To facilitate the interpretation of the rejection

Table 3

Theoretical .05 vs. Calculated Rejection Rates

of Methods by Sample Size

Sample Size	Method							
	Bartlett		IPF		Goodman 1		Goodman 2	
	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]
20	.036	5,774	.088	18,191	.002	21,900	.003	21,900
40	.049	12,252	.071	21,233	.012	21,900	.010	21,900
60	.049	15,378	.061	21,695	.018	21,900	.015	21,900
80	.051	17,487	.060	21,820	.022	21,900	.018	21,900
100	.050	18,934	.054	21,876	.025	21,900	.021	21,900

[a]Out of a possible 21,900 tables

rates given in Table 3, the rejection rates of the four methods (Bartlett, IPF, Goodman 1, Goodman 2) were plotted for each sample size (20, 40, 60, 80, 100). Figure 1 contains the graphs for the four methods as the sample size increases from 20 to 100.

Hypothesis 3

Hypothesis 3 is: For a given dimension, the rejection rates of the methods are the same. For this hypothesis, the 109,500 contingency tables were classified according to dimension. There were three dimensions ($3 \times 3 \times 3$, $2 \times 2 \times 3$, $2 \times 2 \times 2$) with each level of dimension containing the same number of tables. Hence, there were 36,500 contingency tables for each dimension.

For each dimension, Table 4 contains the rejection rate and the number of tables actually analyzed of each method. For example, for the 36,500 contingency tables in the $3 \times 3 \times 3$ classification, the Bartlett rejection rate was .034 and 9,052 tables were analyzed. For the same dimension, the IPF rejection rate was .078 and 35,753 tables were analyzed. For a dimension of $3 \times 3 \times 3$, the Goodman 1 rejection rate was .001 and 36,500 tables were analyzed. The Goodman 2 rejection rate was .001 and 36,500 tables were analyzed in the $3 \times 3 \times 3$ classification.

In addition to containing the results of each dimension, Table 4 contains the results for each method as the

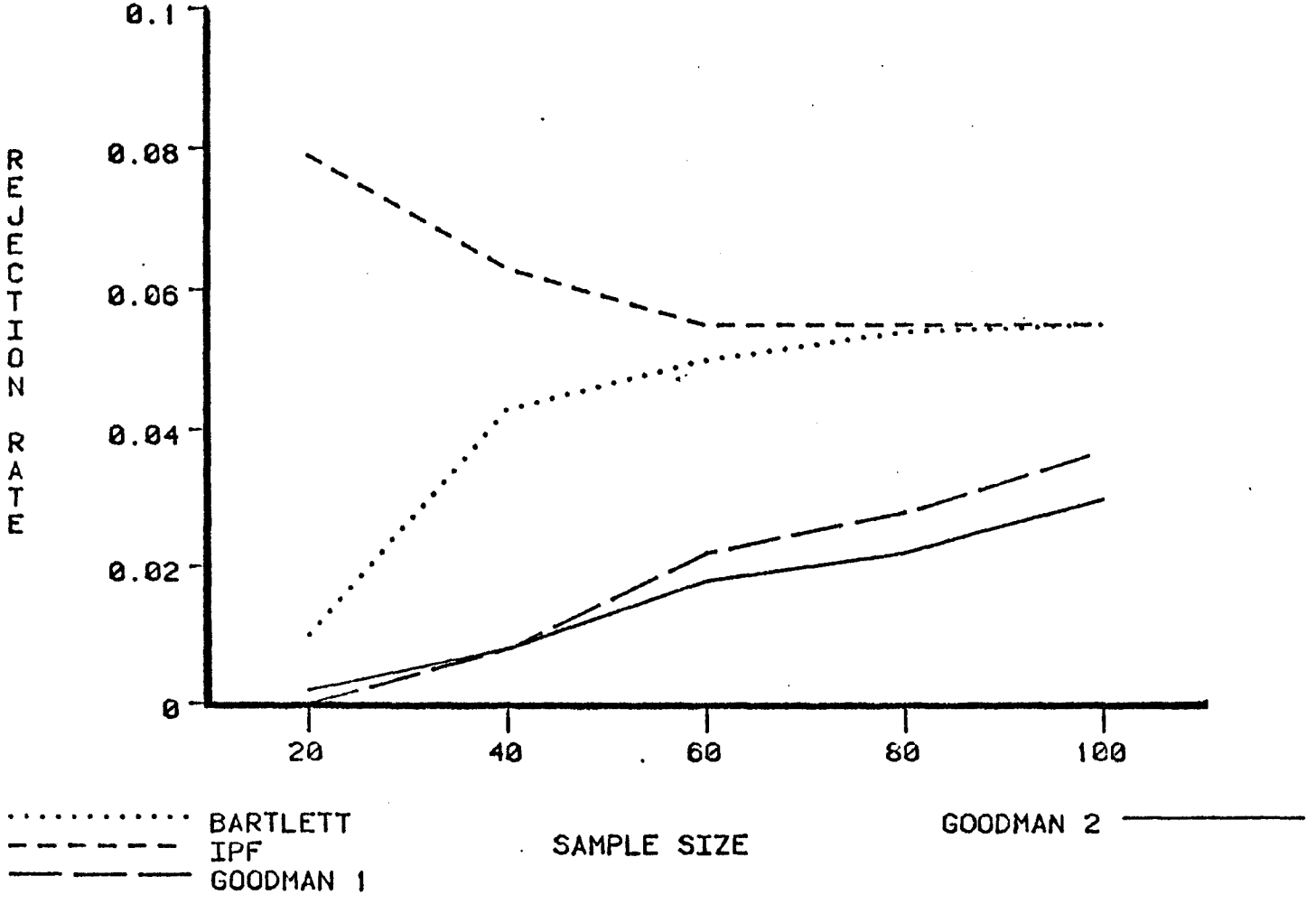


Figure 1. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates by Sample Size

Table 4

Theoretical .05 vs. Calculated Rejection Rates

of Methods by Dimension

Dimension	Method							
	Bartlett		IPF		Goodman 1		Goodman 2	
	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]
3 × 3 × 3	.034	9,052	.078	35,753	.001	36,500	.001	36,500
2 × 2 × 3	.049	28,281	.061	35,656	.019	36,500	.016	36,500
2 × 2 × 2	.053	32,492	.059	33,406	.028	36,500	.024	36,500

[a]Out of a possible 36,500 tables

dimension changes from 3 × 3 × 3 to 2 × 2 × 3 to 2 × 2 × 2. For example, reading down the Bartlett column, for the 3 × 3 × 3 dimension, the Bartlett rejection rate was .034 and 9,052 tables were analyzed, for the 2 × 2 × 3 dimension the rejection rate was .049 and 28,281 tables were analyzed, etc.

To facilitate the interpretation of the rejection rates given in Table 4, the rejection rates of the four methods (Bartlett, IPF, Goodman 1, Goodman 2) were plotted for each dimension (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2). Figure 2 contains the graphs for the four methods as the dimension changes from 3 × 3 × 3 to 2 × 2 × 3 to 2 × 2 × 2.

Hypothesis 4

Hypothesis 4 is: For a given sample size and dimension, the rejection rates of the methods are the same. For this hypothesis, the 109,500 contingency tables were cross-classified according to sample size and dimension. There were five levels of sample size (20, 40, 60, 80, 100) and three levels of dimension (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2). This gave 15 (i.e. 5(3)) combinations of sample size and dimension such that each combination contained the same number of tables (7,300).

Table 5 contains the rejection rate and the number of tables actually analyzed of each method for the 15 combinations of sample size and dimension. For example, in the cross-classification sample size 20 and dimension

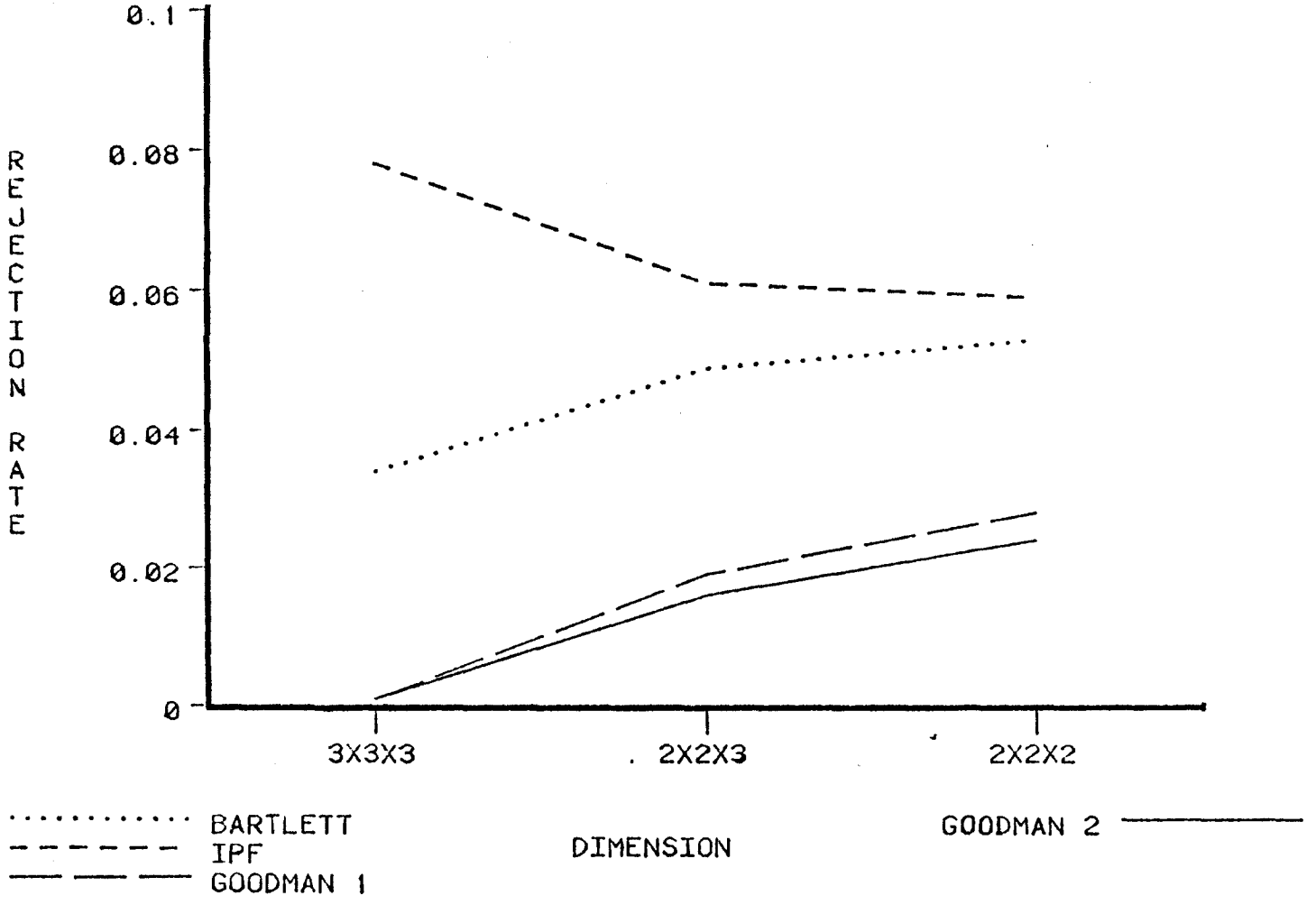


Figure 2. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates by Dimension

Table 5

Theoretical .05 vs. Calculated Rejection Rates

of Methods by Sample Size and Dimension

Sample Size	Method							
	Bartlett		IPF		Goodman 1		Goodman 2	
	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]
				3 × 3 × 3				
20	-	-[b]	.114	6,562	.000	7,300	.000	7,300
40	.000[c]	113	.088	7,292	.000	7,300	.000	7,300
60	.017	1,454	.069	7,299	.000[c]	7,300	.000[c]	7,300
80	.035	3,092	.066	7,300	.001	7,300	.001	7,300
100	.040	4,393	.056	7,300	.003	7,300	.002	7,300
				2 × 2 × 3				
20	.010	1,243	.079	6,509	.000[c]	7,300	.002	7,300
40	.043	5,648	.063	7,249	.008	7,300	.008	7,300
60	.050	6,911	.055	7,298	.022	7,300	.018	7,300
80	.054	7,205	.055	7,300	.028	7,300	.022	7,300
100	.055	7,274	.055	7,300	.037	7,300	.030	7,300
				2 × 2 × 2				
20	.043	4,531	.066	5,120	.006	7,300	.008	7,300
40	.054	6,491	.062	6,692	.027	7,300	.023	7,300
60	.055	7,013	.060	7,098	.032	7,300	.027	7,300
80	.056	7,190	.058	7,220	.038	7,300	.032	7,300
100	.051	7,267	.052	7,276	.036	7,300	.032	7,300

a-Out of a possible 7,300 tables, b-no tables analyzed, c-less than 4 rejected

2 × 2 × 3, the Bartlett rejection rate was .010 and the Bartlett method analyzed 1,243 contingency tables. For the same cross-classification, the IPF rejection rate was .079 and 6,509 contingency tables were analyzed, the Goodman 1 rejection rate was .000 and 7,300 tables were analyzed, and the Goodman 2 rejection rate was .000 and 7,300 tables were analyzed. A "-" for the Bartlett results of sample size 20 and dimension 3 × 3 × 3 means that the Bartlett method failed to analyze any of the 7,300 tables.

In addition to containing the rejection rates for the 15 cross-classifications of sample size and dimension, Table 5 contains the results for other classifications of sample size and dimension. For a given dimension, Table 5 gives the rejection rates for various sample sizes within the given dimension. For example, in the 3 × 3 × 3 classification, the Bartlett rejection rate ranges from nonexistent for the sample size 20 to .040 for the sample size 100. Also, for a given sample size, Table 5 gives the rejection rates for the three levels of dimension. For example, in the classification sample size 20, the Bartlett rejection rates range from nonexistent to .043 as the dimension changes from 3 × 3 × 3 to 2 × 2 × 3 to 2 × 2 × 2.

To facilitate the interpretation of the rejection rates given in Table 5, the rejection rates of the four methods (Bartlett, IPF, Goodman 1, Goodman 2) were plotted

for the five sample sizes (20, 40, 60, 80, 100) within each dimension (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2). Figure 3 contains the graphs for the 3 × 3 × 3 dimension, Figure 4 contains the graphs for the 2 × 2 × 3 dimension, and Figure 5 contains the graphs for the 2 × 2 × 2 dimension.

Hypothesis 5

Hypothesis 5 is: For a given dimension and sampling distribution, the rejection rates of the methods are the same. For this hypothesis, the 109,500 contingency tables were cross-classified according to dimension and sampling distribution. There were three levels of dimensions (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2) and four levels of sampling distribution within a given dimension (see Table 1). This gave 12 combinations of dimension and sampling distribution such that each combination contained the same number of tables (9,125).

Table 6 contains the rejection rate and the number of tables actually analyzed by each method (Bartlett, IPF, Goodman 1, Goodman 2) for the 12 combinations of dimension and sampling distribution. For example, in the cross-classification sampling distribution P-1 (see Table 1) and dimension 3 × 3 × 3, the Bartlett rejection rate was .025 and the Bartlett method analyzed 1,623 tables. For the same cross-classification, the IPF rejection rate was .079 and the IPF method analyzed 8,978 tables, the Goodman

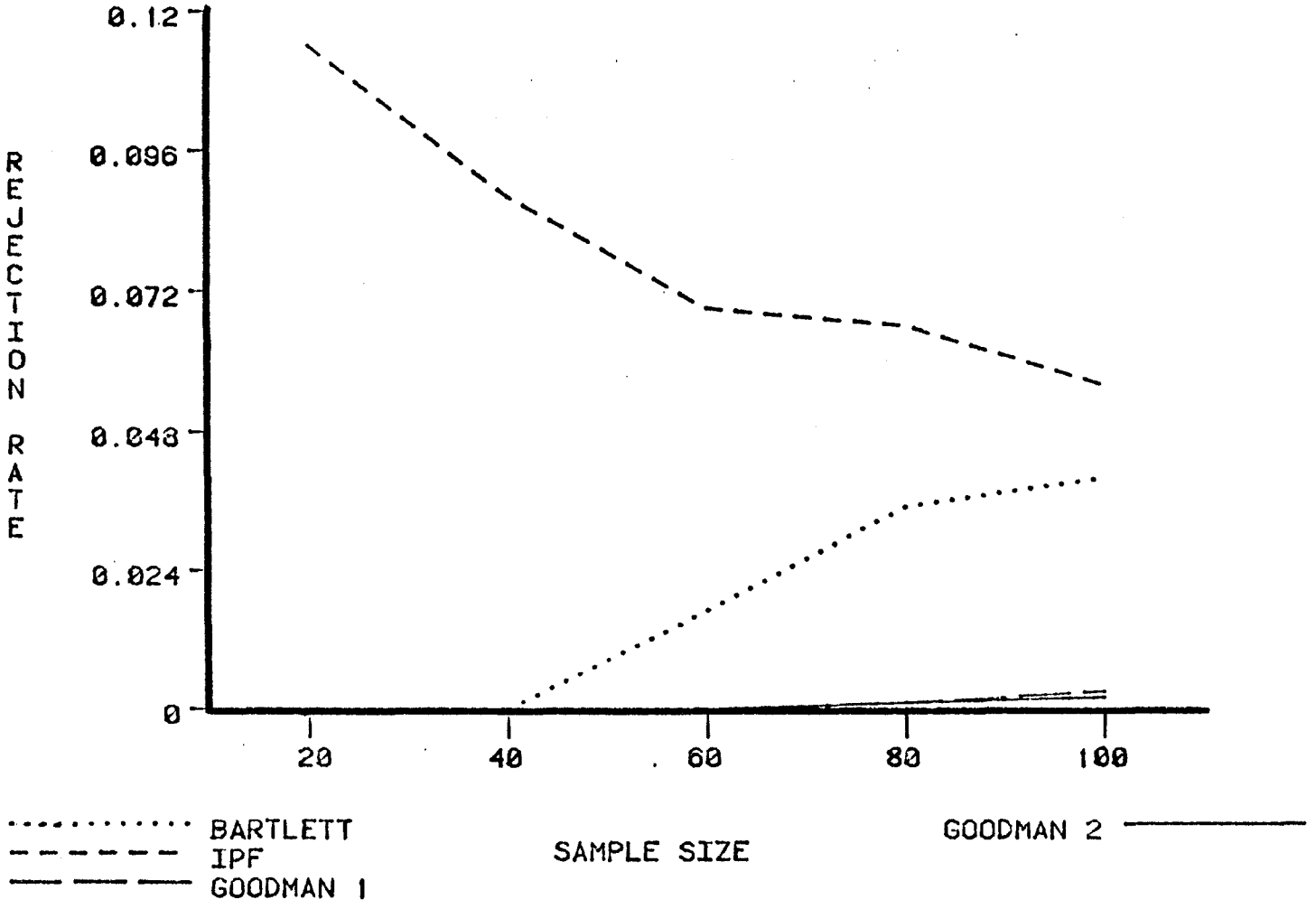


Figure 3. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates for 3 x 3 x 3 Contingency Tables by Sample Size

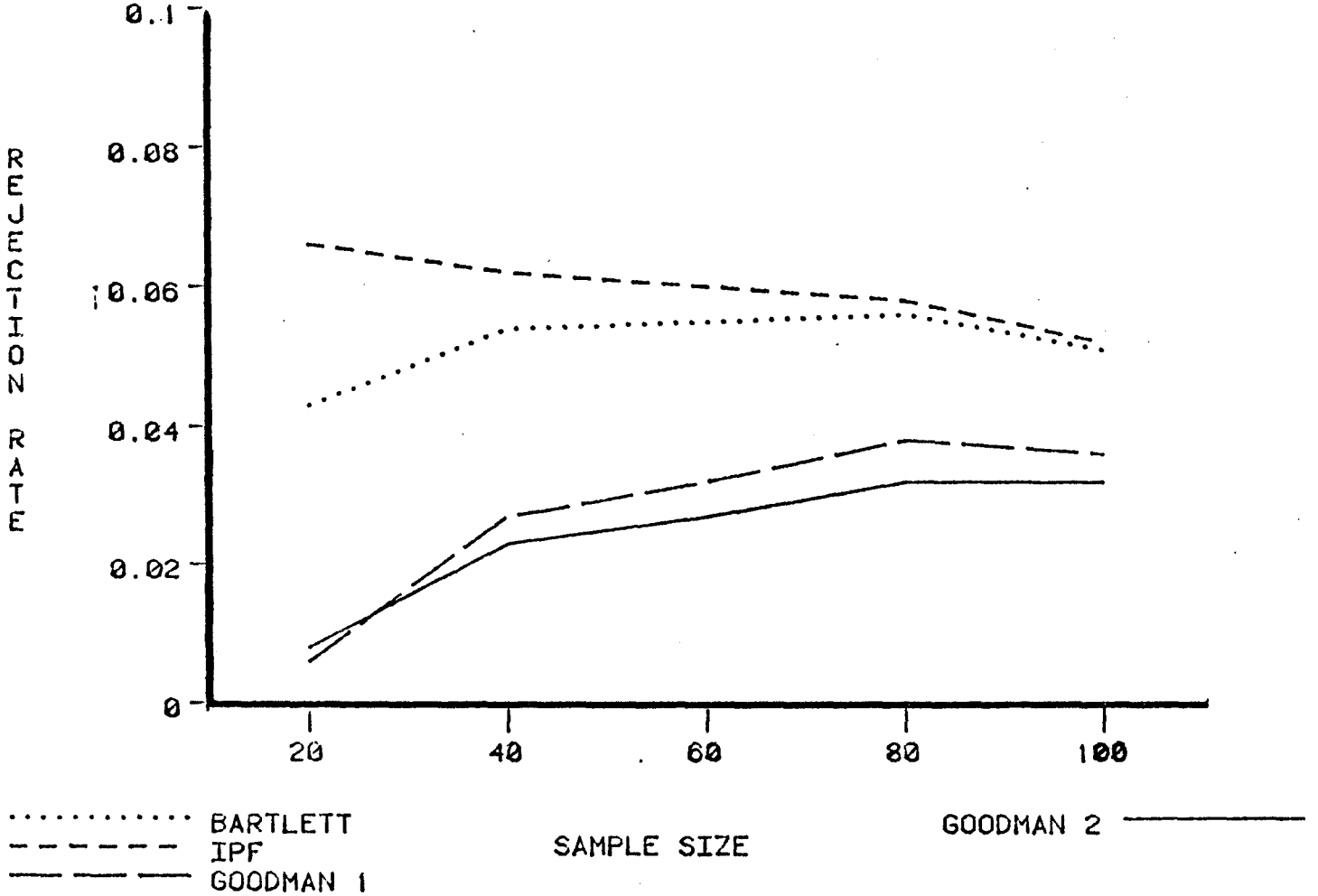


Figure 4. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates for 2 × 2 × 3 Contingency Tables by Sample Size

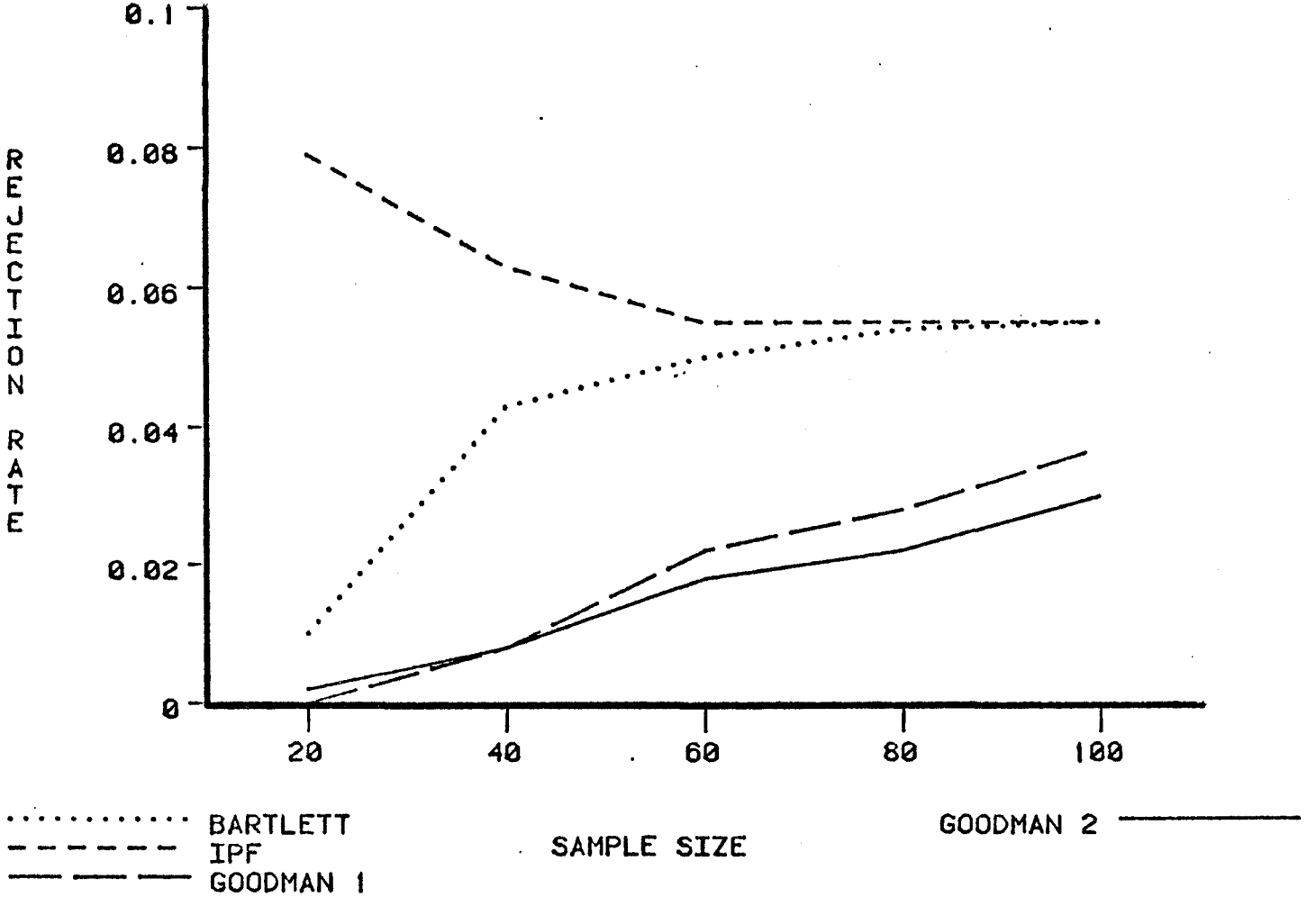


Figure 5. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates for 2 × 2 × 2 Contingency Tables by Sample Size

Table 6

Theoretical .05 vs. Calculated Rejection Rates

of Methods by Sampling Distribution and Dimension

Sampling Distribution[b]	Method							
	Bartlett		IPF		Goodman 1		Goodman 2	
	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]	Rej. Rate	Number Analyzed[a]
				3 × 3 × 3				
P-1	.025	1,623	.079	8,978	.000[c]	9,125	.000[c]	9,125
P-2	.040	4,565	.073	9,087	.003	9,125	.002	9,125
P-3	.008	520	.077	8,663	.000	9,125[c]	.000[c]	9,125
P-4	.034	2,344	.083	9,025	.001	9,125	.001	9,125
				2 × 2 × 3				
Q-1	.045	6,418	.060	8,754	.014	9,125	.013	9,125
Q-2	.043	6,950	.058	8,833	.015	9,125	.013	9,125
Q-3	.055	7,216	.064	9,012	.022	9,125	.017	9,125
Q-4	.052	7,697	.063	9,057	.025	9,125	.020	9,125
				2 × 2 × 2				
R-1	.053	8,427	.057	8,569	.025	9,125	.023	9,125
R-2	.059	8,589	.064	8,738	.036	9,125	.031	9,125
R-3	.036	6,534	.054	7,084	.006	9,125	.007	9,125
R-4	.058	8,942	.061	9,015	.044	9,125	.037	9,125

a- Out of a possible 9,125 tables
b- See Table 1
c- Less than 5 tables rejected

1 rejection rate was .000 (to three decimal places) and the Goodman 1 method analyzed 9,125 tables, and the Goodman 2 rejection rate was .000 (to three decimal places) and the Goodman 2 method analyzed 9,125 tables.

In addition to containing the rejection rates of the methods for the 12 cross-classifications of sampling distribution and dimension, Table 6 contains the rejection rates of the methods (Bartlett, IPF, Goodman 1, Goodman 2) for the sampling distributions within a given dimension. For example, in the 3 × 3 × 3 classification, the Bartlett rejection rates range from .025 to .034 for the sampling distributions P-1 to P-4 (see Table 1). Since the sets of sampling distributions were different for each dimension, no inferences can be made about the rejection rates within a given sampling distribution as the dimension changes from 3 × 3 × 3 to 2 × 2 × 3 to 2 × 2 × 2.

To facilitate the interpretation of the rejection rates given in Table 6, the rejection rates of the four methods (Bartlett, IPF, Goodman 1, Goodman 2) were plotted for the four sampling distributions (see Table 1) within a given dimension. Figure 6 contains the graphs for the 3 × 3 × 3 dimension, Figure 7 contains the graphs for the 2 × 2 × 3 dimension and Figure 8 contains the graphs for the 2 × 2 × 2 dimension.

Hypothesis 6

Hypothesis 6 is: For a given sample size, dimension,

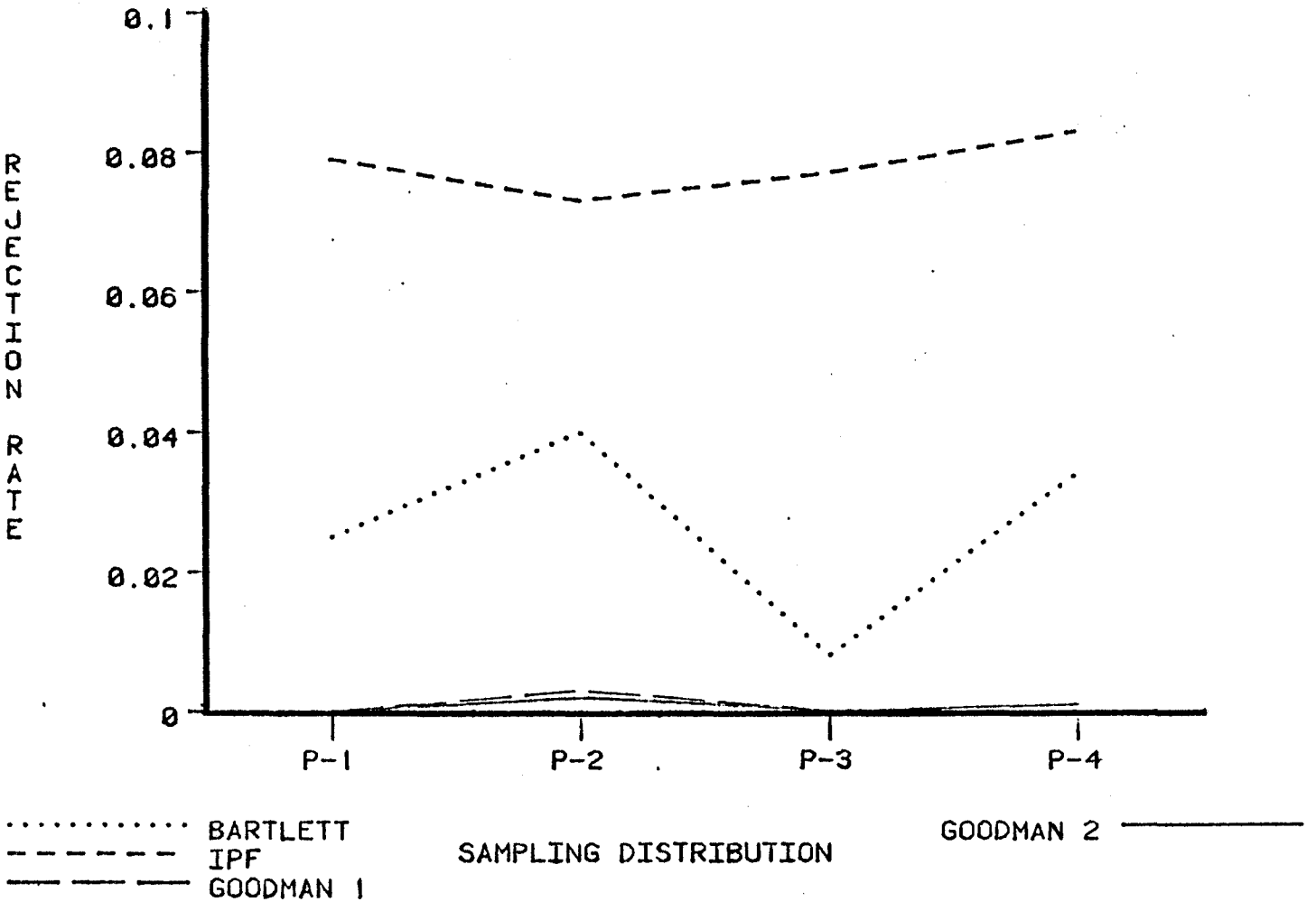


Figure 6. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates for 3 × 3 × 3 Contingency Tables by Sampling Distribution

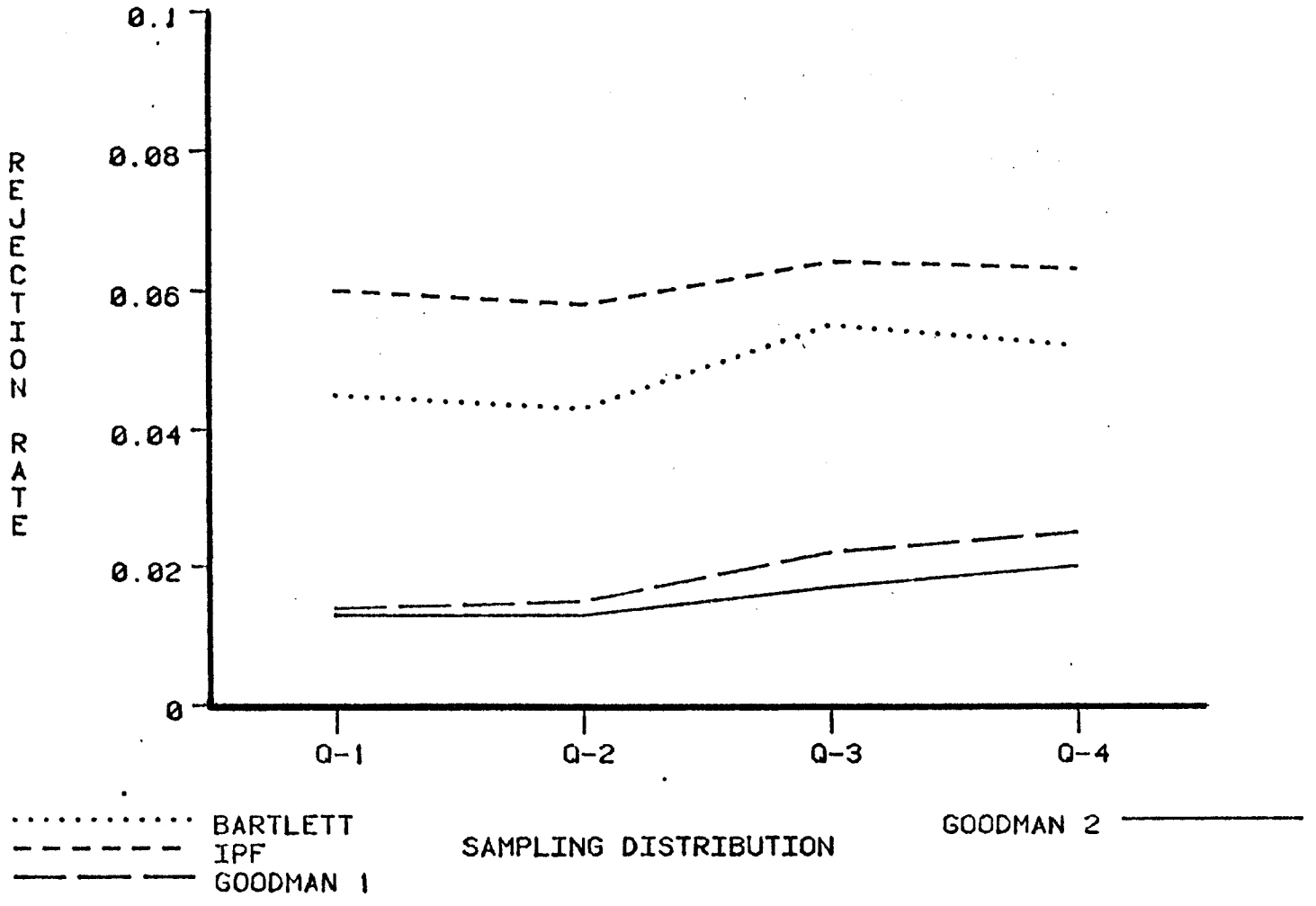


Figure 7. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates for 2 × 2 × 3 Contingency Tables by Sampling Distribution

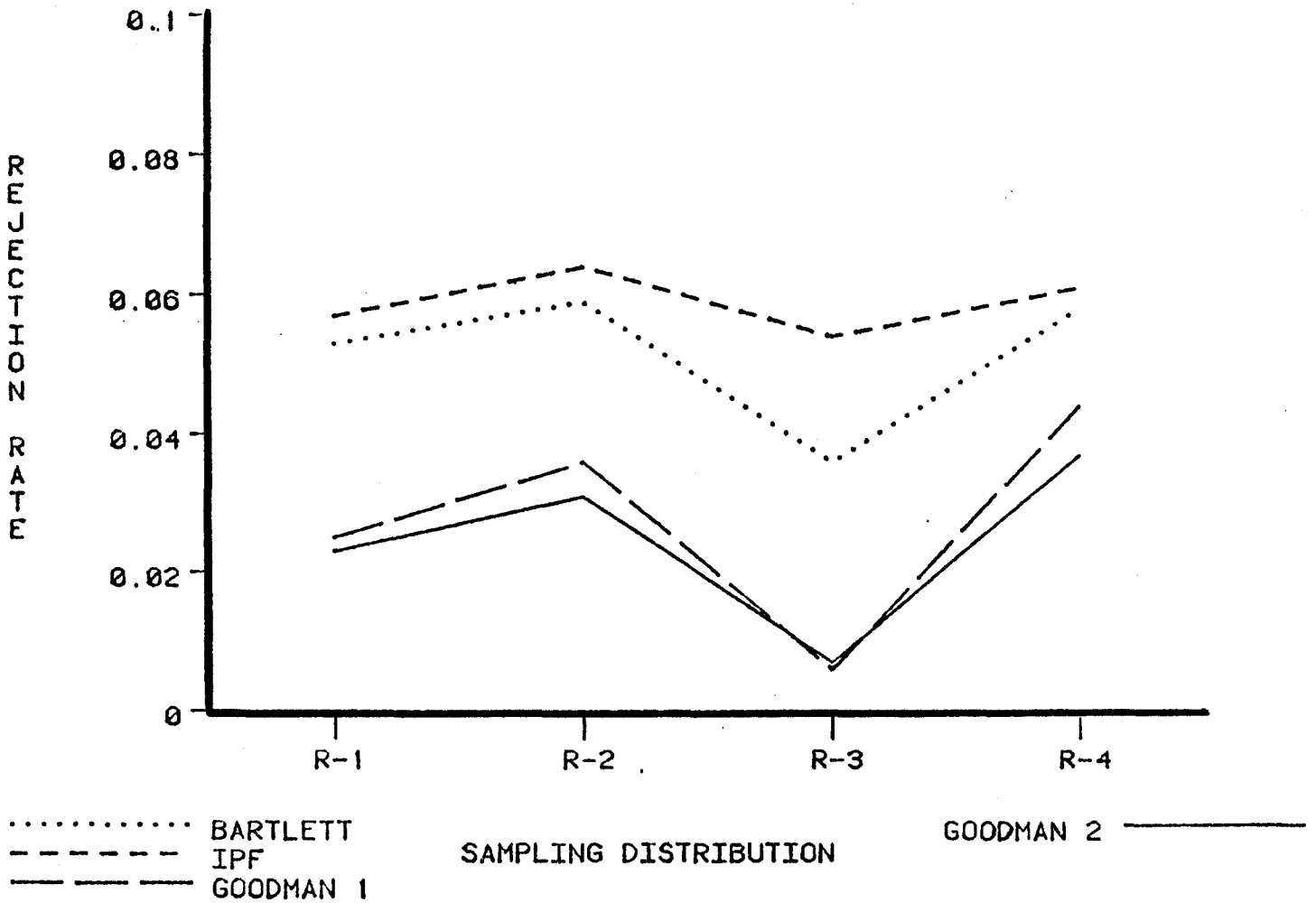


Figure 8. Bartlett, IPF, Goodman 1, and Goodman 2 Rejection Rates for 2 × 2 × 2 Contingency Tables by Sampling Distribution

and sampling distribution, the rejection rates of the methods are the same. This hypothesis takes into account collectively the effects of sample size, dimension, and sampling distribution on the methods (Bartlett, IPF, Goodman 1, Goodman 2).

Tables 7 and 8 contain the rejection rate, the number of tables rejected, and the number of tables analyzed by each method. The data in Tables 7 and 8 came from the original 60 sets of 1,825 contingency tables which were randomly generated for every combination of sample size (20, 40, 60, 80, 100), dimension (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2), and sampling distribution (four per dimension, see Table 1). For example, in Table 7, if the sample size is 100, the dimension is 3 × 3 × 3, and the sampling distribution is P-1, then the Bartlett, IPF, Goodman 1, and Goodman 2 rejection rates were .033, .056, .000, and .001, respectively. At the same time, in Table 8, the number of tables rejected by the Bartlett, IPF, Goodman 1, and Goodman 2 methods were 32 of 957, 102 of 1,825, 1 of 1,825, and 2 of 1,825, respectively.

The data in Tables 7 and 8 were used to calculate the rejection rates, the numbers of tables rejected, and the numbers of tables actually analyzed for Tables 2 through 6. For example, in the Bartlett section of Table 7, if: the numbers of tables rejected were summed up, the numbers of tables actually analyzed were summed

Table 7

Theoretical .05 vs. Calculated Rejection Rates

of Methods by Sample Size and Sampling Distribution within Dimension

Sampling Dist.	Method																			
	Bartlett					IPF					Goodman 1					Goodman 2				
	Sample Size					Sample Size					Sample Size					Sample Size				
	20	40	60	80	100	20	40	60	80	100	20	40	60	80	100	20	40	60	80	100
	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.	Rej.
3 × 3 × 3																				
P-1	-	.000	.007	.015	.033	.111	.082	.076	.071	.056	.000	.000	.000	.000	.000	.000	.000	.000	.001	.001
P-2	-	.000	.020	.045	.049	.112	.083	.059	.060	.052	.000	.000	.002	.003	.008	.000	.000	.001	.003	.004
P-3	-	-	.000	.009	.008	.117	.090	.065	.063	.059	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
P-4	-	.000	.011	.030	.042	.115	.099	.076	.072	.057	.000	.000	.000	.000	.003	.000	.000	.000	.000	.003
2 × 2 × 3																				
Q-1	.000	.029	.038	.057	.054	.076	.066	.044	.060	.055	.000	.003	.014	.023	.031	.000	.005	.014	.021	.026
Q-2	.009	.032	.046	.050	.047	.078	.056	.058	.053	.047	.000	.005	.018	.022	.029	.001	.006	.016	.017	.025
Q-3	.003	.049	.058	.062	.059	.080	.059	.062	.063	.059	.000	.012	.023	.033	.042	.002	.010	.018	.026	.031
Q-4	.017	.056	.057	.046	.061	.082	.069	.059	.046	.061	.000	.013	.033	.033	.045	.004	.010	.026	.025	.037
2 × 2 × 2																				
R-1	.048	.050	.056	.056	.052	.070	.052	.057	.056	.052	.007	.015	.030	.034	.038	.010	.015	.028	.028	.033
R-2	.049	.068	.060	.066	.051	.075	.069	.060	.066	.051	.007	.042	.039	.050	.042	.007	.034	.034	.040	.037
R-3	.011	.022	.035	.039	.049	.055	.060	.057	.047	.053	.000	.004	.005	.009	.014	.002	.005	.007	.010	.012
R-4	.043	.065	.064	.063	.052	.060	.065	.064	.063	.052	.011	.047	.053	.058	.050	.014	.037	.041	.049	.045

Note: 1825 tables per rej. rate. See Table 8 for number of tables rej. and analyzed.

Table 8

Tables Rejected/Tables Analyzed (R/A) of Methods

by Sample Size and Sampling Distribution within Dimension

Sampling Dist.	Method: Bartlett, Sample Size 20 R/A	Bartlett, Sample Size 40 R/A	Bartlett, Sample Size 60 R/A	Bartlett, Sample Size 80 R/A	Bartlett, Sample Size 100 R/A	IPF, Sample Size 20 R/A	IPF, Sample Size 40 R/A	IPF, Sample Size 60 R/A	IPF, Sample Size 80 R/A	IPF, Sample Size 100 R/A
					3 × 3 × 3					
P-1	-	0/4	1/133	8/529	32/957	187/1,678	150/1,825	139/1,825	130/1,825	102/1,825
P-2	-	0/105	21/1,036	74/1,640	88/1,784	200/1,787	152/1,825	108/1,825	109/1,825	95/1,825
P-3	-	-	0/18	1/115	3/387	161/1,372	163/1,817	118/1,824	114/1,825	107/1,825
P-4	-	0/4	3/267	24/808	53/1,265	198/1,725	180/1,825	138/1,825	131/1,825	104/1,825
					2 × 2 × 3					
Q-1	0/162	31/1,086	62/1,616	100/1,751	98/1,803	113/1,484	119/1,796	80/1,824	110/1,825	101/1,825
Q-2	2/217	44/1,381	79/1,720	91/1,811	85/1,821	121/1,552	102/1,807	105/1,824	97/1,825	85/1,825
Q-3	1/331	73/1,484	101/1,758	113/1,818	107/1,825	138/1,716	107/1,821	113/1,825	115/1,825	107/1,825
Q-4	9/533	95/1,697	104/1,817	83/1,825	112/1,825	145/1,757	126/1,825	107/1,825	83/1,825	112/1,825
					2 × 2 × 2					
R-1	57/1,195	89/1,769	102/1,815	102/1,823	95/1,825	92/1,313	93/1,785	103/1,821	102/1,825	95/1,825
R-2	65/1,331	121/1,787	110/1,822	120/1,824	93/1,825	111/1,475	124/1,792	110/1,822	120/1,824	93/1,825
R-3	4/362	24/1,111	54/1,551	67/1,718	88/1,792	34/616	78/1,291	93/1,630	82/1,746	95/1,801
R-4	71/1,643	118/1,824	117/1,825	114/1,825	95/1,825	103/1,716	118/1,824	117/1,825	114/1,825	95/1,825

Note: "-" indicates that no tables were analyzed.

Table 8 (Continued)

Tables Rejected/Tables Analyzed (R/A) of Methods

by Sample Size and Sampling Distribution within Dimension

Sampling Dist.	Method: Goodman 1, Sample Size 20 R/A	40 R/A	60 R/A	80 R/A	100 R/A	Goodman 2, Sample Size 20 R/A	40 R/A	60 R/A	80 R/A	100 R/A
				3 × 3 × 3						
P-1	0	0	0	1	1	0	0	0	2	2
P-2	0	0	3	6	14	0	0	2	6	7
P-3	0	0	0	0	0	0	0	0	0	0
P-4	0	0	0	0	5	0	0	0	0	5
				2 × 2 × 3						
Q-1	0	5	26	42	56	0	9	26	39	48
Q-2	1	9	33	40	53	2	11	29	31	46
Q-3	1	22	41	61	76	4	19	32	47	57
Q-4	1	24	60	60	82	7	19	48	45	67
				2 × 2 × 2						
R-1	12	27	54	62	70	19	28	51	51	61
R-2	12	77	71	92	76	13	62	62	74	68
R-3	1	7	9	16	26	4	9	12	19	22
R-4	20	86	97	105	91	25	68	75	89	82

Note: Goodman 1, 2 analyzed all 1,825 tables. Only number rejected is reported.

up, and the former were divided by the latter, then the result would represent the overall rejection rate of the Bartlett method (.049) which is found in Table 2.

It should be noted that, as in Hypothesis 5, the sets of sampling distributions are different for each dimension (see Table 1). Therefore, the rejection rates for sampling distribution can be compared only within a given dimension.

Results of Univariate Descriptive Statistics

This part of the Results chapter contains the results of the univariate descriptive statistics that were calculated for the Bartlett, IPF, and Goodman (1,2) chi-square statistics (see Equations 1, 38, 46). Specifically, the means, variances, standard errors of the means, and numbers of observations were calculated for these chi-square statistics. The means and variances were compared to the theoretical means and variances of the corresponding chi-square distributions (Hays, 1973).

Since there were five sample sizes (20, 40, 60, 80, 100), three dimensions (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2), and four sampling distributions per dimension (see Table 1), 60 sets of descriptive statistics were calculated. Table 9 contains the means ($\bar{X}$), variances (S^2), standard errors of the means (SE), and the numbers of observations (N) for the 60 combinations of sample size, dimension, and sampling distribution. For example, for the Bartlett method, with a 3 × 3 × 3 dimension, sample size 40 and

Table 9

$\bar{X}$'s, S^2's, SE's, N's of Chi Square Statistics of Methods by Sample Size and Sampling Distribution within Dimension

Sample Size	Sampling Dist.	Method: Bartlett $\bar{X}^d$	Bartlett S^2	Bartlett SE	Bartlett N^a	IPF $\bar{X}^d$	IPF S^2	IPF SE	IPF N^a	Goodman 1[c] $\bar{X}^d$	Goodman 1[c] S^2	Goodman 1[c] SE	Goodman 2[c] $\bar{X}^d$	Goodman 2[c] S^2	Goodman 2[c] SE
						$3 \times 3 \times 3$, $\nu=8$, $2\nu=16$									
20	P-1	-	-	-	- [b]	4.24	11.74	.08	1,678	1.79	.84	.02	2.79	1.66	.03
	P-2	-	-	-	-	6.05	18.68	.10	1,787	2.00	.85	.02	3.25	1.97	.03
	P-3	-	-	-	-	3.58	10.48	.09	1,372	1.66	.73	.02	2.43	1.47	.03
	P-4	-	-	-	-	4.79	12.75	.09	1,725	1.86	.81	.02	2.93	1.67	.03
40	P-1	4.88	.38	.31	4	7.44	17.08	.10	1,825	3.22	2.38	.04	4.07	3.21	.04
	P-2	6.07	6.12	.24	105	8.98	18.36	.10	1,825	3.92	3.03	.04	4.74	3.94	.05
	P-3	-	-	-	-	6.15	16.81	.10	1,817	2.59	2.02	.03	3.45	2.88	.04
	P-4	5.06	1.17	.54	4	8.32	20.16	.11	1,825	3.50	2.85	.04	4.39	3.81	.05
60	P-1	6.98	7.61	.24	133	8.59	20.11	.11	1,825	4.32	3.97	.05	4.90	4.50	.05
	P-2	7.82	11.91	.11	1,036	8.80	15.71	.09	1,825	5.32	5.39	.05	5.50	5.45	.05
	P-3	6.18	4.77	.51	18	7.69	16.01	.09	1,824	3.21	2.71	.04	4.15	3.45	.04
	P-4	7.27	12.30	.21	267	8.64	16.28	.09	1,825	4.58	4.44	.05	5.07	4.87	.05
80	P-1	7.58	9.92	.14	529	8.54	16.11	.09	1,825	4.99	5.45	.06	5.30	5.57	.06
	P-2	8.43	14.95	.10	1,640	8.76	17.13	.10	1,825	6.42	7.28	.06	6.14	6.95	.06
	P-3	7.15	15.28	.36	115	8.32	16.59	.10	1,825	3.86	3.60	.04	4.66	4.22	.05
	P-4	8.07	13.97	.13	808	8.93	17.01	.10	1,825	5.55	6.09	.06	5.74	6.06	.06
100	P-1	8.03	12.24	.11	957	8.66	15.54	.09	1,825	5.62	6.33	.06	5.74	6.18	.06
	P-2	8.39	15.18	.09	1,784	8.44	15.26	.09	1,825	6.82	2.99	.07	6.36	7.28	.06
	P-3	7.46	9.06	.15	387	8.66	14.10	.09	1,825	4.58	4.44	.05	5.17	4.80	.05
	P-4	8.14	13.66	.10	1,265	8.64	15.72	.09	1,825	6.03	7.00	.06	5.98	6.71	.06

a- Out of a possible 1,825 tables
b- A "-" means no tables analyzed
c- All 1825 tables analyzed
d- $p<.003$ for all t-tests on H_o: $\mu=0$

Table 9 (Continued)

$\bar{X}$'s, S^2's, SE's, N's of Chi Square Statistics of Methods by Sample Size and Sampling Distribution within Dimension

Sample Size	Sampling Dist.	Method: Bartlett $\bar{X}$[c]	Bartlett S^2	Bartlett SE	Bartlett N[a]	IPF $\bar{X}$[c]	IPF S^2	IPF SE	IPF N[a]	Goodman 1[b] $\bar{X}$[c]	Goodman 1[b] S^2	Goodman 1[b] SE	Goodman 2[b] $\bar{X}$[c]	Goodman 2[b] S^2	Goodman 2[b] SE
						$2 \times 2 \times 3$, $\nu=2$, $2\nu=4$									
20	Q-1	1.50	1.10	.08	162	2.01	3.51	.05	1,484	.83	.62	.02	1.06	.91	.02
	Q-2	1.55	1.44	.08	217	2.21	5.13	.06	1,552	.89	.74	.02	1.12	1.07	.02
	Q-3	1.69	1.46	.07	331	2.13	4.32	.05	1,716	.98	.83	.02	1.20	1.12	.02
	Q-4	1.73	1.74	.06	533	2.41	5.91	.06	1,757	1.09	.96	.02	1.32	1.29	.03
40	Q-1	1.94	2.45	.05	1,086	2.26	4.34	.05	1,796	1.32	1.34	.03	1.42	1.47	.03
	Q-2	1.98	3.00	.05	1,381	2.24	4.11	.05	1,807	1.39	1.56	.03	1.45	1.59	.03
	Q-3	2.19	3.85	.05	1,484	2.20	4.08	.05	1,821	1.50	1.80	.03	1.51	1.76	.03
	Q-4	2.13	3.71	.05	1,697	2.23	4.46	.05	1,825	1.66	2.15	.03	1.59	1.99	.03
60	Q-1	2.03	3.57	.05	1,616	2.05	3.97	.05	1,824	1.47	1.91	.03	1.47	1.83	.03
	Q-2	2.06	3.78	.05	1,720	2.17	4.40	.05	1,824	1.64	2.22	.03	1.59	2.09	.03
	Q-3	2.16	4.18	.05	1,758	2.18	4.62	.05	1,825	1.76	2.60	.04	1.67	2.39	.04
	Q-4	2.22	4.40	.05	1,817	2.23	4.44	.05	1,825	1.95	2.89	.04	1.80	2.56	.04
80	Q-1	2.18	4.25	.05	1,751	2.20	4.43	.05	1,825	2.18	4.25	.05	1,68	2.35	.04
	Q-2	2.06	3.74	.05	1,811	2.09	3.83	.05	1,825	1.78	2.47	.04	1.67	2.23	.03
	Q-3	2.15	4.27	.05	1,818	2.16	4.32	.05	1,825	1.91	2.94	.04	1.79	2.66	.04
	Q-4	1.97	3.60	.04	1,825	1.97	3.60	.04	1,825	1.84	2.82	.04	1.71	2.50	.04
100	Q-1	2.15	4.34	.05	1,803	2.15	4.47	.05	1,825	1.83	2.74	.04	1.74	2.56	.04
	Q-2	2.09	4.10	.05	1,821	2.09	4.11	.05	1,825	1.85	2.85	.04	1.75	2.61	.04
	Q-3	2.07	4.02	.05	1,825	2.07	4.02	.05	1,825	1.93	3.15	.04	1.81	2.85	.04
	Q-4	2.10	4.24	.05	1,825	2.10	4.24	.05	1,825	2.01	3.53	.04	1.89	3.18	.04

a- Out of a possible 1,825 tables
b- All 1825 tables analyzed
c- $p<.003$ for all t-tests on H_o: $\mu=0$

Table 9 (Continued)

$\bar{X}$'s, S^2's, SE's, N's of Chi Square Statistics of Methods by Sample Size and Sampling Distribution within Dimension

Sample Size	Sampling Dist.	Method: Bartlett $\bar{X}$[c]	Bartlett S^2	Bartlett SE	Bartlett N[a]	IPF $\bar{X}$[c]	IPF S^2	IPF SE	IPF N[a]	Goodman 1[b] $\bar{X}$[c]	Goodman 1[b] S^2	Goodman 1[b] SE	Goodman 2[b] $\bar{X}$[c]	Goodman 2[b] S^2	Goodman 2[b] SE
		$2 \times 2 \times 2$, $\nu=1$, $2\nu=2$													
20	R-1	1.19	1.75	.04	1,195	1.35	2.44	.04	1,313	.55	.55	.02	.63	.68	.02
	R-2	1.20	1.85	.04	1,331	1.39	2.42	.04	1,475	.63	.62	.02	.71	.74	.02
	R-3	.96	.73	.05	362	1.55	1.81	.05	616	.32	.20	.01	.42	.33	.01
	R-4	1.12	1.53	.03	1,643	1.23	2.76	.04	1,716	.70	.77	.02	.73	.82	.02
40	R-1	1.10	1.84	.03	1,769	1.10	1.86	.03	1,785	.76	.96	.02	.76	.92	.02
	R-2	1.21	2.45	.04	1,787	1.21	2.50	.04	1,792	.90	1.35	.03	.88	1.26	.03
	R-3	1.00	1.02	.03	1,111	1.33	2.05	.04	1,291	.44	.37	.01	.57	.54	.02
	R-4	1.09	2.36	.04	1,824	1.09	2.36	.04	1,824	.97	1.63	.03	.89	1.43	.03
60	R-1	1.09	2.30	.04	1,815	1.09	2.36	.04	1,821	.88	1.31	.03	.85	1.22	.03
	R-2	1.13	2.50	.04	1,822	1.13	2.50	.04	1,822	.96	1.57	.03	.91	1.44	.03
	R-3	1.06	1.28	.03	1,551	1.23	2.02	.04	1,630	.58	.56	.02	.67	.67	.02
	R-4	1.09	2.25	.04	1,825	1.09	2.25	.04	1,825	1.04	1.80	.03	.96	1.60	.03
80	R-1	1.03	1.92	.03	1,823	1.03	1.92	.03	1,825	.90	1.40	.03	.86	1.29	.03
	R-2	1.11	2.50	.04	1,824	1.11	2.50	.04	1,824	1.02	1.87	.03	.96	1.72	.03
	R-3	1.07	1.56	.03	1,718	1.13	1.90	.03	1,746	.68	.76	.02	.72	.78	.02
	R-4	1.07	2.25	.04	1,825	1.07	2.25	.04	1,825	1.04	2.01	.03	.98	1.82	.03
100	R-1	1.04	1.99	.04	1,825	1.04	1.99	.03	1,825	.96	1.59	.03	.91	1.45	.03
	R-2	1.02	1.92	.03	1,825	1.02	1.92	.03	1,825	.97	1.59	.03	.91	1.45	.03
	R-3	1.09	1.78	.03	1,792	1.11	1.88	.03	1,801	.78	.92	.02	.78	.90	.02
	R-4	1.00	1.89	.03	1,825	1.00	1.89	.03	1,825	.99	1.77	.03	.94	1.63	.03

a- Out of a possible 1,825 tables
b- All 1825 tables analyzed
c- $p<.003$ for all t-tests on H_o: $\mu=0$

sampling distribution P_1, the mean was 4.88, the variance was .38, the standard error of the mean was .31 and the number of observations was 4. The mean (4.88) and variance (.38) is compared to the theoretical mean of 8 (ν=8) and variance of 16 (2ν=16), where ν represents the degrees of freedom.

In addition to looking at the 60 combinations of sample size, dimension, and sampling distribution, the data were collapsed over sampling distribution. This provided more practical information on the behavior of the methods (Bartlett, IPF, Goodman) than the information found when sampling distribution was included in the breakdown of the chi-square statistics. Table 10 contains the means ($\bar{X}$), variances (S^2), standard errors of the means (SE), and the numbers of observations (N) for the 15 combinations of sample size and dimension (i.e. 5(3)). For example, for the Bartlett method with a 3 × 3 × 3 dimension and sample size 40, the mean was 5.99, the variance was 5.80, the standard error of the mean was .23, and the number of observations was 113. The mean (5.99) and variance (5.80) is compared to the theoretical mean of 8 (ν=8) and variance of 16 (2ν=16).

Results of the Linear Regression Analyses

This part of the Results chapter consists of two sections. In the first section the results of scatter diagrams of the Bartlett, IPF, and Goodman (1, 2)

Table 10

$\bar{X}$'s, S^2's, SE's, N's of Chi Square Statistics of Methods by Dimension and Sample Size

Sample Size	Method											
	Bartlett				IPF				Goodman 1[b]			Goodman 2[b]		
	$\bar{X}$[c]	S^2	SE	N[a]	$\bar{X}$[c]	S^2	SE	N[a]	$\bar{X}$[c]	S^2	SE	$\bar{X}$[c]	S^2	SE
							3 × 3 × 3, ν=8, 2ν=16							
20	-	-	-	-[d]	4.74	14.44	.05	6,562	1.83	.82	.01	2.85	1.78	.02
40	5.99	5.80	.23	113	7.72	19.22	.05	7,292	3.31	2.80	.02	4.16	3.69	.02
60	7.62	11.60	.09	1,454	8.43	17.21	.05	7,299	4.36	4.70	.03	4.90	4.80	.03
80	8.14	13.97	.07	3,092	8.64	16.76	.05	7,300	5.20	6.47	.03	5.46	6.00	.03
100	8.14	13.65	.06	4,393	8.60	15.15	.05	7,300	5.76	7.10	.03	5.81	6.42	.03
							2 × 2 × 3, ν=2, 2ν=4							
20	1.66	1.54	.04	1,243	2.20	4.79	.03	6,509	.95	.80	.01	1.18	1.11	.01
40	2.07	3.34	.02	5,648	2.23	4.25	.02	7,249	1.47	1.73	.02	1.49	1.71	.02
60	2.12	4.00	.02	6,911	2.16	4.36	.02	7,298	1.71	2.43	.02	1.63	2.24	.02
80	2.09	3.97	.02	7,205	2.10	4.05	.02	7,300	1.82	2.69	.02	1.71	2.44	.02
100	2.10	4.17	.02	7,274	2.11	4.21	.02	7,300	1.90	3.07	.02	1.79	2.80	.02
							2 × 2 × 2, ν=1, 2ν=2							
20	1.15	1.62	.02	4,531	1.35	2.47	.02	5,120	.55	.55	.01	.63	.66	.01
40	1.11	2.02	.02	6,491	1.17	2.21	.02	6,692	.77	1.12	.01	.78	1.05	.01
60	1.10	2.11	.02	7,013	1.13	2.29	.02	7,098	.87	1.34	.01	.85	1.24	.01
80	1.07	2.07	.02	7,190	1.08	2.15	.02	7,220	.91	1.53	.01	.88	1.41	.01
100	1.04	1.90	.02	7,267	1.04	1.92	.02	7,276	.92	1.47	.01	.89	1.36	.01

a- Out of a possible 1,825 tables
b- All 1,825 tables analyzed
c- $p<.0001$ for all t-tests on H_o: $\mu=0$
d- No tables analyzed

chi-square statistics (see Equations 1, 38, 46) are given. In the second section, the results of the linear regression analyses are given for the Bartlett, IPF, and Goodman (1, 2) chi-square statistics. The scatter diagrams were used to determine the number of lines to be fitted by the linear regression procedure.

Scatter Diagrams

The chi-square statistics of the Bartlett, IPF, Goodman 1 and Goodman 2 methods were collapsed over sampling distribution. This provided more practical information on the behavior of these methods than if sampling distribution were included with dimension and sample size in the cross-classification of the data. Since there were five sample sizes (20, 40, 60, 80, 100) and three dimensions ($3 \times 3 \times 3$, $2 \times 2 \times 3$, $2 \times 2 \times 2$), there were 15 combinations of sample size and dimension. Scatter diagrams were constructed for five combinations of the Bartlett, IPF, Goodman 1, and Goodman 2 chi-square statistics. Four combinations were Bartlett vs. Goodman 1, IPF vs. Goodman 1, Bartlett vs. Goodman 2, and IPF vs. Goodman 2. The fifth combination was Bartlett vs. IPF.

Depending on its shape, each scatter diagram was placed into one of three categories. The first category consisted of those scatter diagrams which indicated that a single linear relationship existed between the two statistics. The second category consisted of those

scatter diagrams which indicated that two linear relationships existed between the two statistics. The third category consisted of those scatter diagrams which indicated that three linear relationships existed between the two statistics. The following is a breakdown that shows which scatter diagrams fell into which category.

First Category

For Bartlett vs. Goodman 1 and IPF vs. Goodman 1, the following combinations of dimension and sample size had scatter diagrams that fell into the first category. For the 3 × 3 × 3 dimension, the sample sizes of 20, 40, 60, 80, and 100 indicated one linear relationship between the chi-square statistics. For the 2 × 2 × 3 dimension, the sample sizes of 20, 40, 60, 80, and 100 indicated one linear relationship.

For Bartlett vs. Goodman 2 and IPF vs. Goodman 2, the following combinations of dimension and sample size had scatter diagrams that fell into the first category. For the 3 × 3 × 3 and 2 × 2 × 3 dimensions, the sample sizes of 20, 40, 60, 80, and 100 had scatter diagrams that indicated one linear relationship between the chi-square statistics. For the 2 × 2 × 2 dimension and sample sizes 20 and 40, the scatter diagrams indicated that one linear relationship existed.

For Bartlett vs. IPF, every combination of dimension and sample size indicated one linear relationship between

the two chi-square statistics. It should be noted that since the Bartlett method failed to analyze 3 × 3 × 3 tables with sample size 20 (see Table 5) there were no scatter diagrams whenever the Bartlett statistic was used for the 3 × 3 × 3 dimension and sample size 20.

The scatter diagrams which fell into this first category (as well as the other categories) can be found in Appendices A, B, and C. Appendix A contains the scatter diagrams for Bartlett vs. IPF. Appendix B contains the scatter diagrams for Bartlett vs. Goodman 1 and IPF vs. Goodman 1. Appendix C contains the scatter diagrams for Bartlett vs. Goodman 2 and IPF vs. Goodman 2.

The "note" at the bottom of each scatter diagram contains two pieces of information about the given scatter diagram. First, the number of missing values is given. This number represents the number of points that were not plotted because one or both of the statistics which made up the coordinates of the points were missing. These values were missing because either or both methods failed to analyze the tables. Second, the number of hidden points is given. This number represents the number of points having the same coordinates but in excess of 26 points. For example, if there was one point having a given set of coordinates, then that point was plotted with an "A" at its location. If there were two points having the same coordinates, then a "B" was plotted at their

location. Since this particular manner of plotting can provide for only a maximum of 26 points with the same coordinates (i.e. "Z"), then the hidden observation value indicates that some "Zs" reflect more than 26 points that have the same coordinates. Finally, due to the scaling of the axes, the "same coordinates" may mean that the computer was unable to distinguish among points that were relatively close to one another even though their coordinates were not exactly the same.

Second Category

For Bartlett vs. Goodman 1 and IPF vs. Goodman 1, the scatter diagrams of the 2 × 2 × 2 dimension and sample sizes 20 through 100 (Appendix B, Figures 42-51) had scatter diagrams which indicated that two linear relationships existed between the chi-square statistics. For the Bartlett vs. Goodman 2 and IPF vs. Goodman 2, the scatter diagrams of the 2 × 2 × 2 dimension and sample sizes 60, 80, and 100 (Appendix C, Figures 75-80) had scatter diagrams which indicated that two linear relationships existed between the chi-square statistics.

Third Category

For the Bartlett vs. Goodman 1 and IPF vs. Goodman 1, the scatter diagrams of the 2 × 2 × 2 dimension and sample sizes 20 and 40 (Appendix B, Figures 42-45) indicated that two or three possible linear relationships existed between the chi-square statistics. Therefore,

these particular combinations of dimension and sample size were included in both the Second Category and the Third Category of the present chapter.

Linear Regression

This section of the Results chapter contains the results of the linear regression analyses conducted on the Bartlett, IPF, Goodman 1, and Goodman 2 chi-square statistics. The data from these chi-square statistics were cross-classified by dimension and sample size in order to be consistent with the cross-classification used in the Scatter Diagram section of the present chapter. Although a majority of the scatter diagrams indicated that one linear relationship existed between the two chi-square statistics (e.g. see Appendix B, Figure 36: $2 \times 2 \times 3$ and sample size 60), there were 16 scatter diagrams which involved the Goodman statistic that indicated two or three linear relationships (e.g. Appendix B, Figure 42: $2 \times 2 \times 2$ and sample size 20). The following procedure was used to determine the reason for the multiple linear relationships.

An interval on the horizontal axis (i.e. Bartlett or IPF axes) was chosen which contained segments of the multiple lines. The chi-square statistics of the Bartlett (or IPF) method within that interval were printed along with the corresponding Goodman statistics and the corresponding contingency tables. Table 11

Table 11

Bartlett, Goodman 1 Statistics, and Corresponding

2 × 2 × 2 Contingency Tables for Sample Sizes 20 and 40

Table #	Bartlett	Goodman 1	Layer 1 Row 1		Layer 1 Row 2		Layer 2 Row 1		Layer 2 Row 2	
					20					
1	5.36	2.78	0	3	5	3	1	1	1	6
2	5.36	2.78	1	3	5	3	1	1	0	6
3	5.36	2.78	1	0	1	5	1	3	6	3
4	5.36	2.78	0	6	5	3	1	1	1	3
5	5.37	2.23	0	2	2	0	1	3	1	11
6	5.44	3.56	5	1	0	2	2	5	3	2
7	5.49	3.82	1	1	1	5	1	7	3	1
8	5.49	2.21	0	1	2	0	3	4	1	9
9	5.56	2.78	0	3	4	0	2	4	2	5
10	5.60	3.69	1	3	2	1	2	0	2	9
11	5.60	3.69	2	1	2	9	0	2	3	1
12	5.60	3.69	1	3	2	2	2	0	1	9
13	5.63	3.69	0	4	4	1	2	2	2	5
14	5.63	3.69	0	5	2	1	4	2	2	4
15	5.66	2.85	2	0	0	4	1	2	6	5
16	5.67	3.73	0	2	4	3	3	1	1	6
17	5.67	3.73	0	3	2	3	4	1	1	6
18	5.67	3.73	2	1	0	3	1	4	6	3
19	5.67	3.73	2	3	0	4	1	6	3	1
20	5.67	3.73	3	0	1	2	1	4	6	3
					40					
1	5.73	5.16	8	3	7	8	5	18	16	15
2	5.76	5.19	8	7	1	16	4	13	12	16
3	5.77	3.62	0	9	8	12	6	8	11	26
4	5.78	3.59	0	10	6	18	10	9	7	20
5	5.80	5.30	4	7	15	5	10	13	7	19
6	5.82	4.81	1	9	8	9	8	10	9	26
7	5.82	5.35	4	6	19	8	12	10	6	15
8	5.85	3.63	5	0	4	9	13	16	15	18
9	5.86	4.84	1	10	7	11	7	12	5	27
10	5.88	5.20	3	6	4	10	12	9	2	34

contains a partial list of the Bartlett and Goodman 1 statistics and their corresponding contingency tables for the 2 × 2 × 2 dimension and sample sizes of 20 and 80. The data were printed for the Bartlett chi-squares in the closed interval 5.00 to 6.00.

Based upon a number of such print outs, the following procedure was adopted. If a scatter diagram indicated two linear relationships between the chi-square statistics, then one linear regression analysis was done on the chi-square statistics which had no cells with frequencies of zero in the corresponding contingency tables. A second regression analysis was done on the chi-square statistics which had at least one cell with a frequency of zero. If a scatter diagram indicated three linear relationships, then: one linear regression analysis was done on those chi-square statistics which had no cells with frequencies of zero in the corresponding contingency tables, a second linear regression analysis was done on those chi-square statistics which had only one cell with a frequency of zero, and a third linear regression was done on those chi-square statistics which had more than one cell with frequencies of zero.

For those linear regression analyses using Goodman 1 or Goodman 2, the predictor was either Goodman 1 or Goodman 2 and the dependent variable was either Bartlett or IPF. This was done because the Goodman method analyzed

every table (see Table 2) and the results of the linear regression might lead to the formation of a "correction factor" for the Goodman method. For the same reason, since the IPF method analyzed more tables than did the Bartlett method (see Table 2), the predictor was the IPF statistic and the dependent variable was the Bartlett statistic.

One Line

All scatter diagrams of Bartlett vs. IPF indicated one linear relationship existed between these two statistics (see Appendix A). Table 12 contains the results of the linear regression analysis for IPF on Bartlett. The statistics computed for the 15 combinations of dimension and sample size were: the estimated intercept (a) and the estimated slope (b) of the regression lines, their respective standard errors of estimate (SE_a, SE_b), their respective t-values (T_a, T_b) under the null hypotheses, H_0: $\mu_a=0$, H_0: $\mu_b=0$, their respective probabilities (P_a, P_b) of obtaining t-values greater than the obtained t-values under the assumptions that the null hypotheses are true, the sum of squares for the regression model (SS model), the sum of squares for error (SS Error), the degrees of freedom for error (df Error), the F-value (F) for the linear model under the null hypothesis, H_0: $\rho^2=0$, the probability (P_F) of getting an F-value greater than the obtained F-value under the assumption that the null

Table 12

Linear Regression of IPF on Bartlett

with Predictor IPF and Dependent Bartlett

Sample Size	a	SE_a	T_a	P_a	b	SE_b	T_b [a]	SS Model	SS Error	df Error	F [a]	R^2
						3 × 3 × 3						
20	-	-	-	-	-	-	-	-	-	-	-	- [b]
40	-.15	.16	-.94	.3500	1.04	.03	40.13	608.13	41.92	112	1,610.16	.936
60	-.01	.07	-.19	.8456	1.01	.01	114.93	15,181.38	1,668.81	1,452	13,209.02	.901
80	-.03	.04	-.71	.4786	1.01	.00	243.15	41,041.55	2,145.09	3,090	59,120.44	.950
100	.02	.01	1.39	.1654	1.00	.00	705.87	59,415.01	523.62	4,391	99,999.99	.991
						2 × 2 × 3						
20	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	1,908.58	0.00	1,241	99,999.99	1.000
40	0*	.00	.82	.4112	1.00	.00	99,999.99	18,853.23	0*	5,646	99,999.99	1.000
60	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	27,631.95	0.00	6,909	99,999.99	1.000
80	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	28,576.81	0.00	7,203	99,999.99	1.000
100	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	30,363.59	0.00	7,272	99,999.99	1.000
						2 × 2 × 2						
20	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	7,344.34	0.00	4,529	99,999.99	1.000
40	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	13,090.97	0.00	6,489	99,999.99	1.000
60	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	14,824.44	0.00	7,011	99,999.99	1.000
80	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	14,849.44	0.00	7,188	99,999.99	1.000
100	.00	.00	99,999.99	.0000	1.00	.00	99,999.99	13,779.64	0.00	7,266	99,999.99	1.000

* zero to 5 decimal places

[a] $p<.0001$

[b] no tables analyzed by the Bartlett method

hypothesis for the linear model is true, and R-Square (R^2).

Table 13 contains the results of the tests of normality of residuals which resulted from the linear models given in Table 12. The statistics computed for the tests of normality were: the mean of the residuals ($\bar{X}$), the corresponding standard deviation (SD) and the corresponding standard error of estimate (SE), the t-value (T) under the null hypothesis, H_0: $\mu_{res}=0$, the probability (P_T) of getting a t-value greater than the obtained t-value under the assumption that the null hypothesis is true, the D-value (D) under the null hypothesis, H_0: The residuals form a random sample from a normal distribution, the probability (P_D) of getting a D-value greater than the obtained D-value under the assumption that the residuals form a random sample from a normal distribution, and the number (N) of observations. In the test for normality, the null hypothesis is rejected for large values of D (Stephens, 1974; Statistical Analysis System, 1979).

The scatter diagrams of the 3 × 3 × 3 and 2 × 2 × 3 dimensions with the five sample sizes (20, 40, 60, 80, 100) indicated one linear relationship existed between Bartlett (or IPF) and Goodman 1 (or Goodman 2) (Appendix B, Figures 23-41; Appendix C, Figures 52-70). The scatter diagrams of the 2 × 2 × 2 dimension with sample sizes 20 and 40 indicated that one linear relationship existed

Table 13

Tests of Normality for Bartlett Residuals

Resulting from Linear Regression of IPF on Bartlett

Sample Size	$\bar{X}$	SD	SE	T[b]	D	P_D	N
			3 × 3 × 3				
20	-	-	-	-	-	-	-[a]
40	-1.246E-15	.61	.06	-2.165E-14	.42	.01	113
60	1.275E-16	1.07	.03	4.537E-15	.48	.01	1,454
80	-1.863E-15	.83	.01	-1.244E-13	.48	.01	3,092
100	1.920E-16	.35	.01	3.686E-14	.48	.01	4,393
			2 × 2 × 3				
20	0.00	.00	.00	*[c]	*	*	1,243
40	5.282E-18	.00	.00	2.110E-12	.49	.01	5,648
60	0.00	.00	.00	*	*	*	6,911
80	0.00	.00	.00	*	*	*	7,205
100	0.00	.00	.00	*	*	*	7,274
			2 × 2 × 2				
20	0.00	.00	.00	*	*	*	4,531
40	0.00	.00	.00	*	*	*	6,491
60	0.00	.00	.00	*	*	*	7,013
80	0.00	.00	.00	*	*	*	7,190
100	0.00	.00	.00	*	*	*	7,267

a No tables were analyzed by the Bartlett method
b p=1.00 for all T-tests on H_o: $\mu_{res}=0$
c Values are nearly zero

between Bartlett (or IPF) and Goodman 2 (Appendix C, Figures 71-74). Table 14 contains the results of the tests of significance for the intercepts (a) and the slopes (b) of the regression lines. These statistics are the same as the statistics reported in one part of Table 12. Table 15 contains the results of the tests of significance of the linear models given in Table 14. These, too, are the same statistics as those reported in one part of Table 12. Table 16 contains the results of the tests of normality of the residuals which resulted from the linear models given in Table 14. Once again, these statistics are the same as the statistics reported in Table 13. Because the IPF and Bartlett statistics agreed so well for the 2 × 2 × 3 dimension with sample sizes 60, 80, and 100 and for the 2 × 2 × 2 dimension with sample sizes 20, 40, 60, 80, and 100 (see Tables 12, 13), the regression analysis with predictors Goodman 1 and Goodman 2 was done for only the IPF chi-square statistic as the dependent variable.

Two Lines

The scatter diagrams of Bartlett (or IPF) on Goodman 1 for dimension 2 × 2 × 2 with sample sizes 20, 40, 60, 80, and 100 indicated that two linear relationships existed between the two chi-square statistics (Appendix B, Figures 42-51). The scatter diagrams of Bartlett (or IPF) on Goodman 2 for dimension 2 × 2 × 2 with sample

Table 14

Linear Regression with Dependents Bartlett and IPF

and Predictors Goodman 1, Goodman 2 for One Linear Relationship

Sample Size		Goodman 1							Goodman 2						
		a	SE_a	T_a	P_a	b	SE_b	T_b[a]	a	SE_a	T_a	P_a	b	SE_b	T_b[c]
								3 × 3 × 3							
20	IPF	-.38	.08	-4.75	.0001	2.75	.04	71.25	-2.16	.07	-33.06	.0001	2.34	.02	115.77
	Bartlett	-	-	-	-	-	-	-		-	-	-	-	-	- b
40	IPF	.76	.07	11.25	.0001	2.11	.02	115.63	-.81	.05	-15.06	.0001	2.05	.01	173.99
	Bartlett	.84	.26	3.27	.0014	1.70	.08	21.73	.02	.19	.12	.9052	1.73	.05	33.05
60	IPF	1.59	.06	25.45	.0001	1.57	.01	122.38[d]	-.02	.05	-.32	.7510	1.72	.01	187.31
	Bartlett	.82	.10	8.18	.0001	1.41	.02	73.85	.07	.08	.78	.4375	1.54	.02	97.00
80	IPF	1.60	.06	27.06	.0001	1.35	.01	132.50	.15	.04	3.57	.0004	1.55	.01	215.66
	Bartlett	.46	.07	6.65	.0001	1.33	.01	119.97	-.15	.05	-2.71	.0068	1.47	.01	167.83
100	IPF	1.31	.05	24.09	.0001	1.26	.01	147.59	.18	.04	4.75	.0001	1.45	.01	242.52
	Bartlett	.40	.05	7.96	.0001	1.27	.01	170.05	-.10	.03	-2.82	.0049	1.40	.01	261.14
								2 × 2 × 3							
20	IPF	.22	.02	10.00	.0001	2.01	.02	124.14	-.14	.02	-8.56	.0001	1.88	.01	191.55
	Bartlett	.39	.02	17.51	.0001	1.54	.02	75.27	.14	.01	13.23	.0001	1.58	.01	180.43
40	IPF	.13	.02	8.75	.0001	1.42	.01	184.86	-.04	.01	-4.34	.0001	1.52	.01	300.11
	Bartlett	.13	.01	11.04	.0001	1.32	.01	215.58	.01	.01	.70	.4819	1.43	.00	357.86
60	IPF	.00	.01	.00	.8804	1.26	.01	247.38	-.06	.01	-7.05	.0001	1.36	.00	358.27
80	IPF	-.05	.01	-5.45	.0001	1.19	.00	317.50	-.06	.01	-8.58	.0001	1.26	.00	428.12
100	IPF	-.07	.01	-9.36	.0001	1.14	.00	387.24	-.06	.01	-9.53	.0001	1.21	.00	478.86
								2 × 2 × 2							
20									.05	.01	5.36	.0001	1.66	.01	211.41

a $p<.0001$ for all T-tests on H_o: $\mu_b=0$ b No tables were analyzed by the Bartlett method

c $p<.0001$ for all T-tests on H_o: $\mu_b=0$ d $p<.0011$ for T-tests on H_o: $\mu_b=0$

Table 15

Tests of Significance for Corresponding

Linear Models in Table 14

Sample Size		Goodman 1 SS Model	Goodman 1 SS Error	Goodman 1 F^c	Goodman 1 R^2	df^a Error	Goodman 2 SS Model	Goodman 2 SS Error	Goodman 2 F^c	Goodman 2 R^2
					3 × 3 × 3					
20	IPF	41,321.56	53,399.68	5,076.24	.436	6,560	63,595.25	31,125.99	13,403.10	.671
	Bartlett	-	-	-	-	-	-	-	-	- [b]
40	IPF	90,686.09	49,442.89	13,371.01	.647	7,290	112,932.09	27,196.89	30,270.92	.806
	Bartlett	526.33	123.72	472.21	.810	111	590.09	59.96	1,092.40	.908
60	IPF	84,440.90	41,140.44	14,977.12	.672	7,297	103,960.34	21,621.00	35,086.20	.828
	Bartlett	13,307.00	3,543.20	5,453.20	.790	1,452	14,597.36	2,252.83	9,408.33	.866
80	IPF	86,393.30	35,910.75	17,557.37	.706	7,298	105,715.98	16,588.08	46,510.22	.864
	Bartlett	35,553.56	7,633.07	14,392.71	.823	3,090	38,917.30	4,269.33	28,167.04	.901
100	IPF	82,853.24	27,759.67	21,782.07	.749	7,298	98,402.62	12,210.29	58,814.50	.890
	Bartlett	52,037.08	7,901.55	28,917.70	.868	4,391	56,312.74	3,625.89	68,195.42	.940
					2 × 2 × 3					
20	IPF	21,923.83	9,256.84	15,411.12	.703	6,507	26,483.91	4,696.76	36,691.39	.849
	Bartlett	1,565.61	342.97	5,665.04	.820	1,241	1,838.51	70.08	32,556.56	.963
40	IPF	25,397.13	5,385.85	34,173.45	.825	7,247	28,490.54	2,292.43	90,066.30	.926
	Bartlett	16,810.96	2,042.27	46,475.06	.892	5,646	18,057.12	796.11	99,999.99	.958
60	IPF	28,413.59	3,387.59	61,195.57	.893	7,296	30,090.77	1,710.40	99,999.99	.946
80	IPF	27,586.99	1,997.19	99,999.99	.932	7,298	28,451.34	1,132.85	99,999.99	.962
100	IPF	29,295.90	1,425.75	99,999.99	.954	7,298	29,774.03	947.62	99,999.99	.969
					2 × 2 × 2					
20	IPF					5,118	11,361.29	1,301.00	44,694.18	.897

a df. error is the same for Goodman 1, Goodman 2 b No tables analyzed by the Bartlett method

c All Fs have p-values less than .0001

Table 16

Tests of Normality of Residuals

for Linear Models in Table 14

Sample Size		Goodman 1 $\bar{X}$	SD	SE	T[a]	D[b]	N[c]	Goodman 2 $\bar{X}$	SD	SE	T[a]	D[b]
						3 × 3 × 3						
20	IPF	-4.678E-13	2.85	.04	-1.328E-11	.04	6,562	-4.520E-13	2.18	.03	-1.681E-11	.06
	Bartlett	-	-	-	-	-	-	-	-	-	-	- d
40	IPF	-3.487E-13	2.60	.03	-1.143E-11	.06	7,292	-3.790E-13	1.93	.02	-1.676E-11	.10
	Bartlett	2.317E-15	1.05	.10	2.343E-14	.09[e]	113	1.503E-15	.73	.07	2.184E-14	.15
60	IPF	-1.917E-13	2.37	.03	-6.897E-12	.07	7,299	-2.565E-13	1.72	.02	-1.273E-11	.11
	Bartlett	-1.225E-14	1.56	.04	-2.990E-13	.10	1,454	-3.766E-14	1.25	.03	-1.153E-12	.19
80	IPF	-1.044E-13	2.22	.03	-4.020E-12	.06	7,300	-1.921E-13	1.51	.02	-1.089E-11	.08
	Bartlett	-6.916E-14	1.57	.03	-2.447E-12	.07	3,092	-1.197E-13	1.18	.02	-5.662E-12	.12
100	IPF	-9.004E-13	1.95	.02	-3.945E-12	.06	7,300	-1.534E-13	1.29	.02	-1.013E-11	.06
	Bartlett	-8.063E-14	1.34	.02	-3.984E-12	.06	4,393	-1.217E-13	.91	.01	-8.875E-12	.06
						2 × 2 × 3						
20	IPF	-2.100E-14	1.19	.01	-1.421E-12	.10	6,509	-1.241E-13	.85	.01	-1.178E-11	.14
	Bartlett	-7.430E-15	.53	.01	-4.985E-13	.20	1,243	-9.404E-15	.24	.01	-1.396E-12	.10
40	IPF	-6.439E-14	.86	.01	-6.360E-12	.11	7,249	-9.661E-14	.56	.01	-1.463E-11	.10
	Bartlett	-8.988E-15	.60	.01	-1.123E-12	.12	5,648	-2.678E-14	.38	.00	-5.359E-12	.09
60	IPF	-5.548E-14	.68	.01	-6.956E-12	.24	7,298	-6.450E-14	.48	.01	-1.138E-11	.15
80	IPF	-4.532E-14	.52	.01	-7.403E-12	.32	7,300	-6.438E-14	.39	.00	-1.396E-11	.23
100	IPF	-3.158E-14	.44	.01	-6.104E-12	.34	7,300	-3.831E-14	.36	.00	-9.085E-12	.24
						2 × 2 × 2						
20	IPF						5,120	1.504E-13	.50	.01	2.134E-11	.17

a p=1.00 for all T-tests on H_o: $\mu_{res}=0$ b p=.01 for all D-tests on H_o: $D>D_{obtained}$

c N is the same for Goodman 1 and Goodman 2 d No tables analyzed by the Bartlett method

e p=.04 for D-tests on H_o: $D>D_{obtained}$

sizes 60, 80, and 100 indicated that two linear relationships existed between the two chi-square statistics (see Appendix C, Figures 73-80).

Table 17 contains the results of the tests of significance for the intercepts (a) and the slopes (b) of the regression lines. Table 18 contains the results of the tests of significance for the linear models given in Table 17. Table 19 contains the results of the tests of noramlity of the residuals which resulted from the linear models given in Table 17. The statistics reported in Tables 17, 18, and 19 are the same as the statistics reported in Tables 14, 15, and 16 for the "one linear relationship." Because the IPF and Bartlett chi-square statistics agree so well for these particular combinations of dimension and sample size (see Tables 12, 13), the only dependent variable used for the regression analysis in the present section was the IPF chi-square statistic.

Three Lines

The scatter diagrams of Bartlett (or IPF) on Goodman 1 for dimension 2 × 2 × 2 and sample sizes 20 and 40 indicated that there may be two or even three linear relationships between the two chi-square statistics (see Appendix B, Figures 42-45). Therefore, these two combinations of dimension and sample size were re-analyzed under the assumption that three linear relationships existed between the chi-square statistics rather than two linear

Table 17

Linear Regression for 2 × 2 × 2 Tables with Dependent IPF and Predictors Goodman 1, Goodman 2 with Two Linear Relationships

Sample Size	# Empty Cells	Goodman 1						Goodman 2					
		a	SE_a	T_a^a	b	SE_b	T_b^a	a	SE_a	T_a^a	b	SE_b	T_b^a
40	0	-.04	.00	-18.15	1.12	.00	630.74	-.03	.00	-12.40	1.22	.00	651.94
	>0	.58	.02	38.07	1.57	.01	137.71	.28	.01	25.82	1.54	.01	219.10
60	0	-.03	.00	-22.53	1.09	.00	1,082.22	-.02	.00	-10.42	1.17	.00	950.42
	>0	.57	.02	28.04	1.61	.01	126.14	.24	.02	15.17	1.59	.01	180.34
80	0	-.03	.00	-22.93	1.09	.00	1,165.53	-.02	.00	-9.51	1.14	.00	1,035.56
	>0	.56	.02	32.03	1.58	.01	133.34	.26	.01	21.77	1.56	.01	212.71
100	0	-.03	.00	-18.41	1.07	.00	1,171.29	-.01	.00	-5.55	1.12	.00	980.07
	>0	.58	.02	37.26	1.56	.01	159.66	.28	.01	25.11	1.54	.01	241.00

a $p<.0001$ for all T-tests on H_o: $\mu_a=0$ and on H_o: $\mu_b=0$

Table 18

Tests of Significance for Corresponding

Linear Models in Table 17

Sample Size	# Empty Cells	Goodman 1 SS Model	Goodman 1 SS Error	Goodman 1 F[a]	Goodman 1 R^2	df[b] Error	Goodman 2 SS Model	Goodman 2 SS Error	Goodman 2 F[a]	Goodman 2 R^2
40	0	7,146.82	83.98	99,999.99	.988	4,675	7,152.13	78.67	99,999.99	.989
	>0	5,464.47	580.04	18,964.03	.904	2,013	5,801.24	243.27	48,003.90	.960
60	0	9,174.53	45.64	99,999.99	.995	5,826	9,161.08	59.09	99,999.99	.994
	>0	4,969.74	396.03	15,911.92	.926	1,268	5,164.42	201.35	32,522.73	.962
80	0	12,026.86	56.99	99,999.99	.995	6,437	12,011.74	72.10	99,999.99	.994
	>0	2,238.81	98.10	17,778.94	.958	779	2,297.36	39.55	45,245.22	.983
100	0	11,630.55	57.64	99,999.99	.995	6,799	11,606.03	82.15	99,999.99	.993
	>0	1,409.76	26.16	25,490.82	.982	473	1,424.32	11.60	58,081.05	.992

a All Fs have p-values less than .0001

b df. error is the same for Goodman 1, Goodman 2

Table 19

Tests of Normality of Residuals

for Linear Models in Table 17

Sample Size	# Empty Cells	Goodman 1						Goodman 2					
		$\bar{X}$	SD	SE	T[a]	D[b]	N[c]	$\bar{X}$	SD	SE	T[a]	D[b]	
40	0	-3.587E-15	.13	.00	-1.830E-12	.34	4,677	-4.961E-15	.13	.00	-2.616E-12	.17	
	>0	-8.375E-15	.54	.01	-7.005E-13	.28	2,015	-9.878E-15	.35	.01	-1.276E-12	.21	
60	0	1.039E-14	.09	.00	8.965E-12	.31	5,828	2.743E-14	.10	.00	2.079E-11	.13	
	>0	-8.120E-15	.56	.02	-5.180E-13	.30	1,270	-1.072E-14	.40	.01	-9.591E-13	.22	
80	0	2.413E-15	.09	.00	2.058E-12	.31	6,439	2.054E-14	.11	.00	1.558E-14	.17	
	>0	-7.719E-15	.35	.01	-6.083E-13	.24	781	-9.436E-15	.23	.01	-1.171E-12	.15	
100	0	3.324E-15	.09	.00	2.978E-12	.33	6,801	1.722E-14	.11	.00	1.292E-11	.20	
	>0	-2.135E-15	.23	.01	-1.980E-13	.22	475	-4.039E-15	.16	.01	-5.627E-13	.10	

a p=1.00 for all T-tests on H_o: $\mu_{res}=0$

b p=.01 for all D-tests on H_o: $D>D_{obtained}$

c N is the same for Goodman 1 and Goodman 2

relationships (see Two Lines section of the present chapter). Table 20 contains the results of linear regression analyses for these combinations of dimension and sample size. Table 20 is divided into three sections. The first section (Tests of Significance for Intercepts and Slopes) contains the results of the tests of significance for the intercepts (a) and slopes (b) of the linear relationships. The statistics in this first section are the same as those reported in earlier tables (see Tables 12, 14, 17). The second section of Table 20 (Tests of Significance for Linear Models) contains the results of the tests of significance for the linear models given in the first section of Table 20. The statistics given in the second section of Table 20 are the same as those statistics found in earlier tables (see Tables 12, 15, 18). The third section of Table 20 (Tests of Normality of Residuals) contains the results of the tests of normality of the residuals that resulted from the linear models given in the first section of Table 20. The statistics given in the third section of Table 20 are the same as the statistics given in earlier tables (see Tables 13, 16, 19). Because the IPF and Bartlett chi-square statistics agreed so well for these particular combinations of dimension and sample size (see Tables 12, 13), the only dependent variable used for the regression analysis was the IPF statistic.

Table 20

Linear Regression for 2 × 2 × 2 Tables with Dependent IPF and Predictor Goodman 1 with Three Linear Relationships

Tests of Significance for Intercepts and Slopes

Sample Size	# Empty Cells	a	SE_a	T_a	P_a	b	SE_b	T_b	P_b
20	0	-.03	.00	-15.64	.0001	1.11	.00	597.50	.0001
	1	.45	.01	52.84	.0001	1.47	.01	177.48	.0001
	>1	.88	.06	13.58	.0001	2.09	.04	49.17	.0001
40	0	-.04	.00	-18.15	.0001	1.12	.00	630.74	.0001
	1	.52	.01	58.05	.0001	1.50	.01	219.28	.0001
	>1	1.34	.08	15.99	.0001	1.88	.05	35.12	.0001

Tests of Significance for Linear Models

Sample Size	# Empty Cells	SS Model	SS Error	df Error	F	P_F	R^2
20	0	1,026.76	4.97	1,728	99,999.99	.0000	.995
	1	3,739.44	322.91	2,720	31,499.23	.0001	.921
	>1	3,404.78	937.94	666	2,417.63	.0001	.784
40	0	7,146.82	83.98	4,675	99,999.99	.0000	.988
	1	4,453.00	168.65	1,821	48,081.80	.0001	.964
	>1	755.70	116.40	190	1,233.55	.0001	.867

Tests of Normality of Residuals

Sample Size	# Empty Cells	$\bar{X}$	SD	SE	T	P_T	D	P_D	N
20	0	-2.977E-15	.05	.00	-2.310E-12	1.00	.29	.01	1,730
	1	-5.761E-15	.34	.01	-8.724E-13	1.00	.22	.01	2,722
	>1	-1.768E-14	1.19	.05	-3.854E-13	1.00	.32	.01	668
40	0	-3.587E-15	.13	.00	-1.830E-12	1.00	.34	.01	4,677
	1	-7.523E-15	.30	.01	-1.056E-12	1.00	.19	.01	1,823
	>1	1.598E-14	.78	.06	2.837E-13	1.00	.27	.01	192

Unanalyzed Tables

The Goodman method analyzed all 109,500 contingency tables (see Table 2), the IPF method analyzed 96% of the 109,500 contingency tables (see Table 2), and the Bartlett method analyzed 64% of the 109,500 contingency tables (see Table 2). Therefore, the results of unanalyzed contingency tables pertain only to the Bartlett and IPF methods. It should be noted that there were no tables analyzed by the Bartlett method that were not analyzed by the IPF method.

Frequencies of Zero

The Bartlett method will fail to analyze a contingency table whenever the series given in Equation 36 fails to converge. The IPF method will fail to analyze a contingency table whenever the adjusted degrees of freedom (see Equation 101) is less than one. In either case, the number of cells with frequencies of zero may cause either method to fail. Therefore, for each contingency table, the number of cells that had frequencies of zero was calculated. Then the contingency tables were cross-classified according to dimension, sample size, and analyzed vs. not analyzed. As in the Linear Regression section of the present chapter, this particular method of cross-classification is more practical than if sampling distribution were included in the cross-classification.

Table 21 contains the results of this particular

Table 21

Frequency Distribution of Unanalyzed Tables for Bartlett and IPF Methods with respect to Number of Zero Cell Frequencies by Dimension and Sample Size

Sample Size			Number of Empty Cells																		
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3 × 3 × 3																						
20	B	A[a]								0	0	0	0	0	0	0	0	0	0	0	0	0
		NA[b]								3	33	138	423	929	1,484	1,646	1,332	802	371	110	24	5
	IPF	A													1,478	1,615	1,185	556	173	28	1	0
		NA													6	31	147	246	198	82	23	5
40	B	A	5	23	46	34	5	0	0	0	0	0	0	0	0	0	0	0				
		NA	0	2	37	238	531	928	1,079	1,192	1,091	843	592	406	179	54	14	1				
	IPF	A														51	10	0				
		NA														3	4	1				
60	B	A		543	472	111	7	0	0	0	0	0	0	0	0	0						
		NA		67	485	1,065	1,164	985	840	562	351	204	79	33	8	3						
	IPF	A													7							
		NA													1							
80	B	A	995	1,023	498	91	2	0	0	0	0	0	0	0	0							
		NA	1	203	712	1,067	908	624	370	189	91	31	8	2	2							
	IPF	A																				
		NA																				
100	B	A	1,540	1,154	465	63	0	0	0	0	0	0	0									
		NA	3	281	780	807	565	276	124	48	20	2	1									
	IPF	A																				
		NA																				

a "A" refers to the number of tables analyzed for the given number of empty cells

b "NA" refers to the number of tables not analyzed for the given number of empty cells

Note: Missing data indicate that there were either no tables with the given number of empty cells or all tables with the given number of empty cells were analyzed

Table 21 (Continued)

Frequency Distribution of Unanalyzed Tables for Bartlett and IPF Methods with respect to Number of Zero Cell Frequencies by Dimension and Sample Size

Sample Size			Number of Empty Cells						
			1	2	3	4	5	6	7
			2 × 2 × 3						
20	B	A	924	174	0	0	0	0	0
		NA	1	1,932	2,309	1,352	394	64	5
	IPF	A			2,223	952	148	10	0
		NA			86	400	246	54	5
40	B	A	2,675	67	0	0	0		
		NA	126	1,187	286	52	1		
	IPF	A			259	29	0		
		NA			27	23	1		
60	B	A	1,699	16	0	0			
		NA	81	288	19	1			
	IPF	A			18	0			
		NA			1	1			
80	B	A	895	6	0				
		NA	21	71	3				
	IPF	A							
		NA							
100	B	A	470	1					
		NA	13	13					
	IPF	A							
		NA							
			2 × 2 × 2						
20	B	A	2,702	99	0	0	0		
		NA	20	1,764	790	191	4		
	IPF	A		650	18	0	0		
		NA		1,213	772	191	4		
40	B	A	1,811	3	0	0			
		NA	12	617	167	13			
	IPF	A		192	0	0			
		NA		428	167	13			
60	B	A	1,185	0	0				
		NA	5	260	22				
	IPF	A		80	0				
		NA		180	22				
80	B	A	751	0	0				
		NA	2	99	9				
	IPF	A		28	0				
		NA		71	9				
100	B	A	466	0					
		NA	1	32					
	IPF	A		8					
		NA		24					

method of cross-classification. In the second column of Table 21 a "B" or "IPF" refers to the Bartlett or IPF methods respectively. In the third column of Table 21, an "A" or "NA" refers to contingency tables that were analyzed or were not analyzed. The columns headed by the numbers 1 to 20 refer to the number of cells that had frequencies which were zero. Since some tables might not have any cells with frequencies that were zero and since the maximum number of cells is 27 (i.e. 3 × 3 × 3 dimension), the theoretical range for these column headings is 0 to 27. However, it was not necessary to report the results for the entire range. The main concern of this particular section was the contingency tables not analyzed by the Bartlett and IPF methods. Therefore, it was not necessary to report the data for those numbers in the range of 0 to 27 when either there were no tables that fell into that classification (i.e. number of cells with zero frequencies) or if all the tables in that classification were analyzed.

For example, in the 3 × 3 × 3 dimension and sample size 40 classification, Table 21 shows that the first instance in which the Bartlett method failed to analyze contingency tables occurred when the contingency tables had two cells with frequencies of zero. In this particular case, 23 tables with two cells that had frequencies of zero were analyzed and two tables with two cells that had

frequencies of zero were not analyzed. At the opposite end of this 0 to 27 range, there were no tables that had more than 16 cells with frequencies of zero. Therefore, it was necessary to report only the breakdown of the number of cells with frequencies of zero for the range of 2 to 16. It should be noted that no results were reported for tables that had nonzero frequencies in every cell. This means that either no tables fell into this particular category (e.g. Bartlett for 3 × 3 × 3 and sample size 40) or all tables in this particular category were analyzed (e.g. Bartlett for 3 × 3 × 3 and sample size 60).

Negative Chi-Square Statistics

As discussed in the Method chapter, it is impossible to get negative chi-square statistics for the IPF method. However, the Bartlett method could give negative chi-square statistics because the series found in Equation 36 did not converge (McNamee, 1973). The Goodman method could give negative chi-square statistics when the second term on the right-hand side of Equation 46 is greater than the first term on the right-hand side of Equation 46. For the 109,500 randomly generated contingency tables, neither the Bartlett method nor the Goodman method resulted in negative chi-square statistics.

Non-Positive Degrees of Freedom

As discussed in the Method chapter, the IPF method could result in zero or negative degrees of freedom because

of the adjustment to the degrees of freedom due to random zeros in the cells. For the 109,500 randomly generated contingency tables, 4,663 tables had zero degrees of freedom and 22 tables had negative degrees of freedom. Table 22 contains a sample of these 4,685 tables. The tables are written out according to rows within layers of the tables.

Summary of Results

The purpose of the present study was to investigate the robustness of the Bartlett, IPF, and Goodman methods of testing for second order interaction in R × C × L contingency tables when the samples are small. The primary indicator of robustness was the behavior of the rejection rates of these methods for the null hypothesis, no second order interaction. Rejection rates were computed for various combinations of dimension (3 × 3 × 3, 2 × 2 × 3, 2 × 2 × 2), sample size (20, 40, 60, 80, 100), and sampling distribution (four per dimension). The rejection rates were reported in terms of the six hypotheses given in the Method chapter.

In addition to the rejection rates, univariate statistical results were reported for the individual Bartlett, IPF, and Goodman chi-square statistics. In particular, the means and variances of each method's chi-square statistics were computed for the 60 combinations of dimension, sample size, and sampling distribution, and for

Table 22

IPF Tables with Zero or Negative Degrees of Freedom

Dimension	Table #	df	Layer 1									Layer 2									Layer 3								
			Row 1			Row 2			Row 3			Row 1			Row 2			Row 3			Row 1			Row 2			Row 3		
3 × 3 × 3	1	-1	0	0	0	0	1	0	0	1	0	0	4	0	0	0	0	0	3	2	1	0	2	1	0	1	2	0	2
	2	-1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	3	2	0	2	0	0	1	2	2	0	6
	3	-1	0	0	0	1	0	0	0	1	0	1	0	2	0	0	0	0	0	3	0	0	1	0	1	2	0	0	8
	4	-1	1	0	0	0	0	2	2	0	2	0	1	0	0	0	0	0	0	2	0	0	0	0	0	0	1	2	7
	5	-1	1	0	0	0	0	0	0	2	3	0	0	0	1	0	1	0	0	1	1	0	0	1	1	3	0	1	4
	6	-1	0	0	1	1	0	1	0	0	0	0	1	2	0	0	0	1	1	4	0	0	2	0	0	3	0	0	3
	7	-1	0	0	0	0	3	0	1	0	1	0	0	1	0	4	0	0	0	1	2	0	4	0	0	0	3	0	0
	8	-1	1	0	0	1	0	3	0	0	0	0	0	0	0	0	2	2	1	4	0	1	0	0	0	3	0	0	2
	9	-1	1	0	0	0	0	2	0	0	0	1	2	0	0	0	0	0	1	1	0	1	0	0	2	2	0	1	6
	10	-1	1	0	0	2	0	1	3	0	0	0	1	0	0	0	0	0	0	2	2	1	0	0	2	1	0	0	4
	11	-1	0	0	0	0	0	0	2	3	1	0	0	0	3	0	0	0	3	0	0	0	0	0	0	1	0	3	4
	12	-1	1	0	0	0	0	1	4	0	2	0	1	0	0	0	0	0	0	3	0	0	0	0	1	0	2	3	2
	13	-1	0	1	0	0	0	0	2	1	2	0	0	1	0	0	1	1	0	5	0	1	0	0	1	0	1	3	0
	14	-1	0	2	0	0	0	0	4	3	0	0	0	0	0	0	0	0	3	1	0	0	1	1	0	1	2	0	2
	15	-1	0	2	1	1	1	0	5	0	1	0	2	0	0	0	0	3	0	0	0	0	0	1	0	0	0	0	3
	16	-1	0	0	0	1	1	0	4	0	4	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	6
	17	-1	0	0	0	0	0	1	3	0	5	0	0	1	0	0	0	1	3	0	0	1	0	0	1	0	2	2	0
	18	-1	0	0	0	0	0	1	0	0	6	0	0	0	0	1	0	2	3	0	0	1	0	0	0	0	4	1	1
	19	-2	0	0	0	0	0	0	1	2	3	1	0	3	0	1	0	0	0	0	0	0	0	0	0	0	2	1	6
	20	-1	0	0	4	0	0	1	0	0	1	0	2	1	0	0	2	0	0	0	0	0	0	0	0	1	1	3	4
	21	-1	0	0	2	0	0	0	3	1	1	0	0	0	0	2	1	0	0	0	0	0	4	1	0	1	1	0	3
2 × 2 × 2	22	-1	3	0		0	4					1	0		0	12													
3 × 3 × 3	23	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0	3	0	1	1	0	0	0	1	2	2	0	2	4
	24	0	0	0	1	0	0	0	1	0	2	0	0	1	0	1	0	0	3	1	0	0	2	0	2	0	1	2	3
2 × 2 × 3	25	0	0	1		0	3					0	2		0	1					1	3		0	9				
	26	0	2	2		2	1					0	0		1	4					0	0		4	4				
2 × 2 × 2	27	0	0	3		3	1					0	7		2	4													
	28	0	2	2		0	3					3	7		0	3													

Note: All tables listed have sample size 20.

the 15 combinations of dimension and sample size.

Linear regression results of Goodman on IPF, Goodman on Bartlett, and IPF on Bartlett were reported for the 15 combinations of dimension and sample size. It was hoped that the regression equations would lead to a correction factor that might be used to estimate the Bartlett or IPF chi-square statistic when either one could not be computed by the method itself. In some cases, two or three lines were fitted to the chi-square data.

Finally, a breakdown of the unanalyzed contingency tables of the Bartlett and IPF methods was reported with respect to the number of cells in a given table that had frequencies of zero. These results may serve as a "barometer" to indicate when a particular method would fail to analyze a given contingency table.

In the next chapter, there is a discussion of these results, an application of these results to a small sample contingency table that was found in the literature, the conclusions drawn from the present study, and recommendations for further study.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

This chapter consists of three parts. Part one contains a discussion of the results that were reported in Chapter Four. Part two contains an application of the present study to a small sample study found in the literature. Part three contains recommendations for further study.

Discussion of Results

This part of Chapter Five discusses the results that were reported in Chapter Four. The order of discussion parallels the order of presentation of the results reported in Chapter Four.

Rejection Rates

Six hypotheses that were related to the rejection rates of the Bartlett, Iterative Proportional Fitting (IPF), and Goodman methods were given in the Method chapter and the results of the investigations of these six hypotheses were reported in the Results chapter. The results of the rejection rates were given for the Bartlett, IPF, and two versions of the Goodman method (Goodman 1, Goodman 2). The following is a discussion of the results of the investigations of these six hypotheses.

Hypothesis 1

Null Hypothesis 1: The rejection rates of the methods are the same. This hypothesis was concerned with the overall rejection rates of the methods (Bartlett, IPF, Goodman 1, Goodman 2) when dimension, sample size, and sampling distribution were ignored. The results for this hypothesis were given in Table 2. The Bartlett rejection rate (.049) was closest to the theoretical .05 level of significance. However, the Bartlett method did not analyze as many tables as did the IPF and Goodman methods (69,825 vs. 104,815 and 109,500, respectively).

The IPF method (.066) was the least conservative method. This is to say that the IPF method was most likely to reject H_0, no second order interaction. On the other hand, the Goodman method (.016, .014, respectively) was the most conservative method. This means that the Goodman method was least likely to reject H_0, no second order interaction. Since the rejection rates of Goodman 1 and Goodman 2 were almost equal, the overall Goodman rejection rate was not affected significantly by the two procedures used for calculating the Goodman statistic (i.e. add ½ to empty cells vs. add ½ to every cell).

If a Type I error (falsely rejecting H_0) is the more important error to be guarded against, then the Goodman method is the choice because of its low rejection rate. If a Type II error (falsely accepting H_0) is the more

important error to be guarded against, then the IPF method is the choice because of its high rejection rate.

Therefore, the rejection rates and the number of tables analyzed by the methods were different, and so Hypothesis 1 (The rejection rates of the methods are the same.) is rejected.

Hypothesis 2

Null Hypothesis 2: For a given sample size, the rejection rates of the methods are the same. The results of the rejection rates for this particular hypothesis were given in Table 3.

Across sample sizes, the Bartlett rejection rates (.036 to .050) were consistently closest to the theoretical .05 level of rejection. The Bartlett rejection rates were within .001 of this .05 value for all sample sizes except sample size 20. The discrepancy for sample size 20 (.036) was expected since a sample size this small will result in a significant number of empty cells. This in turn will result in the Bartlett method not analyzing many tables. For example, the Bartlett method could analyze only 5,774 of the possible 21,900 tables with sample size 20. Furthermore, it was not until the sample size reached 60 that the Bartlett method was able to analyze at least 70% of the 21,900 tables in this particular category.

For all sample sizes, the IPF method rejected

consistently H_0 more than any other method (.088, .071, .061, .060, .054). The Goodman method rejected consistently H_0 less than any other method (Goodman 1: .002 to .025; Goodman 2: .003 to .021). This indicates that the IPF method was the least conservative method and the Goodman method was the most conservative method. As in the investigation of Hypothesis 1, there was not much difference between the rejection rates of Goodman 1 and Goodman 2 across sample sizes. In addition, the IPF and Goodman methods analyzed consistently more tables across sample sizes (IPF: 18,191 to 21,876; Goodman: 21,900) than the Bartlett method (5,774 to 18,934).

If guarding against making a Type I error (falsely rejecting H_0) is more important than guarding against making a Type II error (falsely accepting H_0), then the Goodman method is the choice because of its low rejection rates. If on the other hand, it is more important to protect against making a Type II error, then the IPF method is the choice for sample sizes up to 80 because of its high rejection rates. When a contingency table can be analyzed by the Bartlett method, then the Bartlett method gives the most consistent results for sample sizes 40 to 100.

As expected, as the sample size increased, then all methods had rejection rates that approached the theoretical .05 level of significance. However, for even

sample size 100, the Goodman rejection rates (.025, .021) were considerably less than this nominal .05 value.

Therefore, for a given sample size, there were differences among the methods with respect to rejection rates. In addition, for a given method, the sample sizes had expected effects on the rejection rates and the number of tables analyzed. Hence, Hypothesis 2 (For a given sample size, the rejection rates of the methods are the same.) is rejected.

Hypothesis 3

Null Hypothesis 3: For a given dimension, the rejection rates of the methods are the same. The results of the rejection rates for this particular hypothesis were given in Table 4. Across dimensions, the Bartlett rejection rates (.034, .049, .053) were consistently close to the theoretical .05 level of significance. The lowest rejection rate (.034) and the smallest number of tables analyzed (9,052) occurred for the 3 × 3 × 3 dimension. This was due to the large number of empty cells that occurred in these tables.

For all dimensions, the IPF method (.078, .061, .059) rejected consistently H_0, no second order interaction, more than any other method. It was not until the 2 × 2 × 2 dimension that the IPF rejection rate (.059) came close to the Bartlett rejection rate (.053). The Goodman 1 (.001, .019, .028) and Goodman 2 (.001, .016, .024) methods

rejected consistently H_0 less than any other method. This indicated that the IPF method was the least conservative method (rejecting H_0 most often) and the Goodman method was the most conservative method (rejecting H_0 least often). As in the investigations of Hypotheses 1 and 2, there was not much difference between the Goodman 1 and Goodman 2 rejection rates across dimensions. Also, the IPF (33,406 to 35,753) and Goodman (36,500) methods analyzed consistently more tables than the Bartlett method (9,052 to 32,492). In particular, the smallest number of tables analyzed by either the IPF or Goodman methods occurred for the IPF method (33,406) for the 2 × 2 × 2 dimension.

If protecting against making a Type I error (falsely rejecting H_0) is more important than protecting against making a Type II error (falsely accepting H_0), then the Goodman method is the choice because of its low rejection rates. If it is more important to guard against a Type II error, then the IPF method is the choice because of its high rejection rates.

As expected, as the number of cells decreased from 27 (i.e. 3 × 3 × 3 dimension) to 8 (i.e. 2 × 2 × 2 dimension), the rejection rates of all methods approached the nominal 5% level of rejection. However, even for the 2 × 2 × 2 dimension, the Goodman rejection rates (.028, .024) were considerably less than this nominal .05 value.

The biggest change in the rejection rates occurred when the dimension changed from 3 × 3 × 3 to 2 × 2 × 3 (e.g. Bartlett: .034 to .049).

Therefore, for a given dimension, there were considerable differences among the methods. Also, for a given method, the dimensions had an expected effect on the rejection rates and the number of tables analyzed. Hence, Hypothesis 3 (For a given dimension, the rejection rates of the methods are the same.) is rejected.

Hypothesis 4

Null Hypothesis 4: For a given sample size and dimension, the rejection rates of the methods are the same. The results of the rejection rates for this particular hypothesis were given in Table 5. Table 5 reported the combined effects of dimension and sample size on the rejection rates (Bartlett, IPF, Goodman 1, Goodman 2) and indicated the sources of the discrepancies among these rejection rates which were discussed in Hypotheses 2 and 3 of the present chapter.

For the 3 × 3 × 3 dimension, the low Bartlett rejection rate (.034) and the small number of tables analyzed (9,052) in Table 4 are explained by the Bartlett rejection rates and the numbers of tables analyzed in the 3 × 3 × 3 section of Table 5 for sample sizes 20, 40 and 60 (-, -; .017, 113; .017, 1,454). These particular rejection rates and numbers of tables analyzed lowered

the overall Bartlett results for the 3 × 3 × 3 dimension in Table 4. Similarly, in Table 3, the Bartlett rejection rate (.036) and the number of tables analyzed (5,774) for sample size 20 was considerably below the Bartlett rejection rates and the numbers of tables analyzed of the remaining sample sizes. Table 5 showed that these results in Table 3 can be explained by the poor performance of the Bartlett method for the 3 × 3 × 3 and 2 × 2 × 3 dimensions with sample size 20. In these particular cross-classifications, the Bartlett method either failed to analyze any of the tables (3 × 3 × 3) or analyzed only 1,243 of the possible 7,300 tables (2 × 2 × 3).

For Hypothesis 3, the IPF method was characterized as being the least conservative method across dimensions. The results found in Table 5 indicated that this was due primarily to: the 3 × 3 × 3 dimension for sample sizes 20 to 80 (.114 to .066), the 2 × 2 × 3 dimension for sample sizes 20 and 40 (.079, .063), and the 2 × 2 × 2 dimension for sample sizes 20 to 60 (.066 to .060). Similarly, for Hypothesis 2, the IPF method was categorized as being the least conservative method across sample sizes. These high rejection rates (greater than .060 in Table 3) resulted from the interaction of dimension and sample size. For example, in Table 5, the cross-classification 3 × 3 × 3 and sample size 20 resulted in rejection rate of .114. This rejection rate was reduced

to .079 when the dimension changed to 2 × 2 × 3 and was reduced to .088 when the sample size was increased to 40. In addition, the results for Hypotheses 2 and 3 indicated that the IPF method did not analyze all tables (see Tables 3, 4). Table 5 showed that the majority of these unanalyzed tables were accounted for by the sample size 20 across all dimensions (6,562, 6,509, 5,120) and the 2 × 2 × 2 dimension for sample size 40 (6,692).

In the discussion of Hypotheses 2 and 3, the Goodman method was characterized as being the most conservative method and it was reported that there was not much difference between the Goodman 1 and Goodman 2 rejection rates. Table 5 showed that the interaction of dimension and sample size had an effect on the results that were discussed in Hypotheses 2 and 3 of the present chapter. For example, in Table 5, the 3 × 3 × 3 dimension and sample size 100 resulted in rejection rates of .003 and .002 (i.e. Goodman 1, Goodman 2). However when the dimension was changed to 2 × 2 × 3, then the rejection rates increased to .037. In addition, for all cross-classifications of dimension and sample size, the rejection rates of Goodman 1 and Goodman 2 were quite close to each other. This indicated that there was no interaction effect of dimension and sample size on the corresponding rejection rates of Goodman 1 and Goodman 2.

Therefore, the results reported in Table 5 are

supportive and explanatory of the results reported in Tables 3 and 4 for Hypotheses 2 and 3. Since Hypotheses 2 and 3 were rejected, Hypothesis 4 is important because the results of the investigation of Hypothesis 4 specify the differences that were discussed in Hypotheses 2 and 3. Hence, Hypothesis 4 (For a given sample size and dimension, the rejection rates of the methods are the same.) is rejected.

Hypothesis 5

Null Hypothesis 5: For a given dimension and sampling distribution, the rejection rates of the methods are the same. The results of the rejection rates for this particular hypothesis are given in Table 6. Since the sampling distributions were not the same for each dimension (see Table 1), it would not be meaningful to investigate the behavior of the rejection rates for a given sampling distribution across dimensions.

For the 3 × 3 × 3 dimension, the IPF method was consistently the least conservative method (rejected H_0 most often) across sampling distributions (.079, .073, .077, .083). The Bartlett and Goodman methods were consistently the most conservative methods (rejected H_0 least often) across sampling distributions (Bartlett: .008 to .034; Goodman 1 and Goodman 2: .000 to .003). The lowest rejection rate of the Bartlett method (.008) occurred for P-3. P-3 had rows in proportion 2:3:5 and

columns and layers in proportion 6:6:7. At the present time there is no explanation for the discrepancy between the results for P-3 and the other three sampling distributions for the Bartlett method. As with the previous hypotheses, there was not much difference between the Goodman 1 and Goodman 2 corresponding rejection rates.

With respect to Type I error (falsely rejecting H_0) and Type II error (falsely accepting H_0), the Goodman method guards consistently against making a Type I error because of its low rejection rates. On the other hand, the IPF method guards consistently against making a Type II error because of its high rejection rates.

In terms of the number of tables analyzed, the Bartlett method analyzed the smallest number of tables (1,623, 4,565, 520, 2,344) out of a possible 9,125 tables for each sampling distribution. The IPF method analyzed a larger number of tables (8,978, 9,087, 8,663, 9,025) than did the Bartlett method. The Goodman method analyzed all tables (see Table 2).

For the 2 × 2 × 3 dimension, the IPF method was consistently the least conservative method (rejected H_0 most often) across sampling distributions (.060, .058, .064, .063). The Goodman method was consistently the most conservative method (rejected H_0 least often) across sampling distributions (Goodman 1: .014 to .025; Goodman 2: .013 to .020). The Bartlett method was consistently closest to

the theoretical 5% level of rejection (.045, .043, .055, .052). Since the number of cells decreased from 27 (i.e. 3 × 3 × 3) to 12 (i.e. 2 × 2 × 3), it was not unusual to find the rejection rates of all methods closer to this nominal .05 value. As with the previous four hypotheses, there was not much difference between Goodman 1 and Goodman 2.

With respect to Type I error (falsely rejecting H_0) and Type II error (falsely accepting H_0), the Goodman method guards consistently against making a Type I error because of its low rejection rates. The IPF method guards consistently against making a Type II error because of its high rejection rates.

In terms of the number of tables analyzed, the Bartlett method analyzed the smallest number of tables (6,418, 6,950, 7,216, 7,697) out of a possible 9,125 tables for each sampling distribution. The IPF method analyzed more tables (8,754, 8,833, 9,012, 9,057) than did the Bartlett method. The Goodman method analyzed all tables (see Table 2).

For the 2 × 2 × 2 dimension, the IPF method continued to be the least conservative method (rejected H_0 most often) across sampling distributions (.057, .064, .054, .061). The Goodman method continued to be the most conservative method (rejected H_0 least often) across sampling distributions (Goodman 1: .006 to .044; Goodman

2: .007 to .037). The Bartlett rejection rates (.053, .059, .036, .058) were closest to the theoretical 5% level of rejection with the exception of R-3 (.036). At the present time, there seems to be no reason why the R-3 sampling distribution would have a rejection rate so different from the other rejection rates for the 2 × 2 × 2 dimension. As with the previous hypotheses, there was not much difference between the Goodman 1 and Goodman 2 rejection rates.

With respect to Type I error (falsely rejecting H_0) and Type II error (falsely accepting H_0), the Goodman method guards against making a Type I error because of its low rejection rates. The IPF method guards consistently against making a Type II error because of its high rejection rates. However, in both cases the rejection rates were quite close to the .05 level of rejection.

In terms of the number of tables analyzed, the Bartlett method analyzed the smallest number of tables (8,427, 8,589, 6,534, 8,942) out of a possible 9,125 tables for each sampling distribution. The IPF method analyzed more tables (8,569, 8,738, 7,084, 9,015) than the Bartlett method and the Goodman method analyzed all tables (see Table 2).

Therefore, there were considerable differences among the methods for the various sampling distributions within a given dimension. These differences lead to the

rejection of Hypothesis 5 (For a given dimension and sampling distribution, the rejection rates of the methods are the same.)

Hypothesis 6

Null Hypothesis 6: For a given sample size, dimension, and sampling distribution, the rejection rates of the methods are the same. The results of the rejection rates of the methods for this particular hypothesis are reported in Tables 7 and 8. Table 7 contains the rejection rates and Table 8 contains the numbers that were used to calculate these rejection rates. As in Hypothesis 5, the combined effect of dimension, sample size, and sampling distribution may be considered only within a given dimension. The data in Table 8 served as the basis from which the data reported for the investigations of Hypotheses 1 through 5 were generated. Also, the data in Tables 7 and 8 pinpoint the discrepancies among the methods that were discussed in Hypothesis 5.

For the 3 × 3 × 3 dimension, the rejection rates of the methods behaved in a manner that was consistent with the first five hypotheses. Except for P-3 (see Table 1), for each sampling distribution, the Bartlett and Goodman rejection rates increased toward the theoretical 5% level of significance (e.g. For P-1: Bartlett: .000 to .033) as the sample size increased. The IPF rejection rates decreased toward the nominal 5% level of rejection (e.g.

P-1: .111 to .056). The only discrepancy occurred for sample size 20 and the Bartlett method. In this particular case the Bartlett method failed to analyze any tables because of the number of empty cells. Among the four sampling distributions, the Bartlett method behaved poorly for P-3 (see Table 1). Table 8 showed that the Bartlett method failed to analyze any tables for sample sizes 20 and 40 and analyzed very few tables (18, 15, 387 out of a possible 1,825 tables per sample size) for the 60, 80, and 100 sample sizes. This explains the behavior of the Bartlett method for P-3 in Hypothesis 5. At the present time, there is no reason for this discrepancy with P-3. As in the previous hypotheses, there was not much difference between the Goodman 1 and Goodman 2 rejection rates for the 3 × 3 × 3 dimension.

With respect to Type I error (falsely rejecting H_0) and Type II error (falsely accepting H_0), the Goodman method guards consistently against making a Type I error because of its low rejection rates. The IPF method guards consistently against making a Type II error because of its high rejection rates. The Bartlett method came close to the theoretical 5% level of rejection for P-2 with sample sizes 80 and 100 (.045, .049) and for P-4 with sample size 100 (.042). For the remaining cross-classifications of the Bartlett method, the rejection rates were considerably below the nominal 5% value and

this indicates that the Bartlett method was quite conservative for these cross-classifications.

In terms of the number of tables analyzed, Table 8 shows that the Bartlett method came close to analyzing the 1,825 tables in each cross-classification for P-2 with sample sizes 80 and 100 (1,640, 1,784). The smallest number of tables analyzed by the IPF method occurred for P-3 and sample size 20 (1,372). This indicates that the Bartlett method was affected more than the IPF method by the number of empty cells. The Goodman method analyzed all tables (see Table 2).

For the 2 × 2 × 3 dimension, the rejection rates of the Bartlett and IPF methods were very similar to each other (e.g. Q-1 with sample size 80: Bartlett: .057, IPF: .060). The Goodman 1 and Goodman 2 rejection rates were almost the same for all cross-classifications in the 2 × 2 × 3 category. In some cases (e.g. Q-4 with sample size 100: Goodman 1: .045, Goodman 2: .037) the Goodman rejection rates were quite close to the theoretical .05 level of rejection.

With respect to Type I error (falsely rejecting H_0) and Type II error (falsely accepting H_0), the Goodman method guards consistently against making a Type I error because of its low rejection rates. The IPF method had 10 of 15 rejection rates greater than .060. This indicated that the IPF method protected against making a

Type II error. The Bartlett method protected against making a Type I error for sample size 20 (rejection rates from .000 to .017). However, Table 8 showed that for sample size 20, the Bartlett method did not analyze very many tables (162, 217, 331, 533 of 1,825 tables per cross-classification).

When the dimension changed from 3 × 3 × 3 to 2 × 2 × 3, then the number of tables analyzed by the Bartlett and IPF methods increased. Except for Q-1 to Q-3 with sample sizes 20 and 40, the Bartlett method analyzed over 1,600 tables in each cross-classification. This was expected since the number of empty cells in a given 2 × 2 × 3 contingency table should be less than the number of empty cells in a given 3 × 3 × 3 contingency table. In every cross-classification except Q-1 and Q-2 with sample size 20 (1,484, 1,552), the IPF method analyzed over 1,700 tables. The Goodman method analyzed all tables (see Table 2).

The highest degree of consistency among the methods occurred for the 2 × 2 × 2 dimension. Except for the discrepancy which surfaced in R-3 (see the discussion of Hypothesis 5) and the differences for sample size 20 (Bartlett: .048, .049, .011, .043 vs. IPF: .070, .075, .055, .060), the Bartlett and IPF rejection rates were almost the same. Also, except for the sample size 20 (Goodman 1: .000 to .011, Goodman 2: .002 to .014) and R-3 (Goodman 1: .000 to .014, Goodman 2: .002 to .012)

both sets of Goodman rejection rates were considerably closer to the theoretical 5% level of rejection. In addition, corresponding Goodman rejection rates were almost the same.

With respect to Type I error (falsely rejecting H_0) and Type II error (falsely accepting H_0), the Goodman method continued to be the most conservative method (rejected H_0 the least number of times) and guards against making a Type I error because of its low rejection rates. For a Type II error, neither the Bartlett nor the IPF method guards consistently against making a Type II error. The only possible exception may be for sample size 20. In this case the IPF rejection rates (.070, .075, .055, .060) were higher than the corresponding Bartlett rejection rates (.048, .049, .011, .043) and therefore the IPF method guards against making a Type II error.

In terms of tables analyzed, the number of tables analyzed by the Bartlett and IPF methods in the 2 × 2 × 2 dimension increased considerably over the number of tables analyzed in the 2 × 2 × 3 and 3 × 3 × 3 dimensions. This was expected since the number of cells decreased from 12 (i.e. 2 × 2 × 3 dimension) to 8 (i.e. 2 × 2 × 2 dimension) and most tables in the 2 × 2 × 2 dimension would have fewer empty cells than in the 2 × 2 × 3 dimension. The only discrepancy occurred for R-3 (Bartlett: 362 to 1,792 and IPF: 616 to 1,801). The Goodman method analyzed all

tables (see Table 2).

Therefore, the results reported in Tables 7 and 8 indicated that for a given dimension, sample size, and sampling distribution, there were considerable differences among the methods. Hence, Hypothesis 6 (For a given sample size, dimension, and sampling distribution, the rejection rates of the methods are the same.) is rejected.

Univariate Results

A secondary indicator of the robustness of the methods (Bartlett, IPF, Goodman) for small sample sizes was the behavior of the means and variances of the methods' chi-square statistics. If the methods are robust for small samples, their empirical means and variances should be close to the theoretical means and variances of the corresponding chi-square distribution. Empirical chi-square means and variances are considered to be close to the theoretical chi-square means and variances when the empirical values were within 10% of the theoretical values.

Table 9 contains the empirical means and variances for the basic 60 combinations of dimension (3), sample size (5), and sampling distribution (4). The results contained in Table 9 are useful to the extent that they provide information on the behavior of the sampling distributions used to generate the contingency tables that were analyzed in the present study.

Based upon the 10% criterion, the following Bartlett

means and variances were close to the theoretical means and variances. For the 3 × 3 × 3 dimension, sample size 80 and P-3 (7.15, 15.28) and sample size 100 and P-2 (8.39, 15.18) were close to the theoretical mean (8) and variance (16). For the 2 × 2 × 3 dimension with theoretical mean, 2 and variance, 4, sample size 80 and Q-3 and Q-4 and sample sizes 60, 80, and 100 and all sampling distributions had means and variances that satisfied the 10% criterion. In these particular cases, the means ranged from 1.97 to 2.22 and the variances ranged from 3.57 to 4.34. For the 2 × 2 × 2 dimension, sample size 40 and R-1, sample sizes 60 and 80 and R-4, and sample size 100 and all sampling distributions had means and variances that were close to the theoretical mean, 1 and variance, 2. The means for these particular cases ranged from 1.00 to 1.10 and the variances ranged from 1.78 to 2.25.

Based upon the 10% criterion, the IPF method had the following means and variances close to the theoretical means and variances. For the 3 × 3 × 3 dimension, sample size 40 and P-1, sample size 60 and P-2 to P-4, and sample sizes 80 and 100 for all sampling distributions had means and variances close to the theoretical mean, 8 and variance, 16. For these particular cases, the means ranged from 7.44 to 8.93 and the variances ranged from 14.10 to 17.13. For the 2 × 2 × 3 dimension with theoretical mean, 2 and variance, 4, sample size 20 and Q-1 and Q-3, sample size

40 for all sampling distributions, sample size 60 and Q-1, Q-2, Q-4, and sample sizes 80 and 100 for all sampling distribution had means and variances close to the theoretical mean and variance. For these particular cases, the means ranged from 1.97 to 2.26 and the variances ranged from 3.51 to 4.47. For the 2 × 2 × 2 dimension, sample size 40 and R-1, sample size 60 and R-4, sample size 80 and R-1, R-3, R-4 and sample size 100 for all sampling distributions had means and variances close to the theoretical mean, 1 and variance, 2. For these particular cases, the means ranged from 1.00 to 1.13 and the variances ranged from 1.86 to 2.27.

Based upon the 10% criterion, the Goodman method had the following means and variances close to the theoretical mean and variance. For the 3 × 3 × 3 dimension, no Goodman 1 and Goodman 2 means and variances were close to the theoretical mean, 8 and variance, 16. For the 2 × 2 × 3 dimension, Goodman 1 for sample size 80 and Q-1 (2.18, 4.25) and sample size 100 and Q-4 (2.01, 3.53) had means and variances close to the theoretical mean, 2 and variance, 4. No Goodman 2 means and variances were close to 2 and 4 for the 2 × 2 × 3 dimension. For the 2 × 2 × 2 dimension, Goodman 1 had the following means and variances close to the theoretical mean, 1 and variance, 2: sample size 60 and R-4 (1.04, 1.80), sample size 80 and R-2 (1.02, 1.87), sample size 80 and R-4 (1.04, 2.01), and sample size 100

and R-4 (.99, 1.77). The Goodman 2 mean (.98) and variance (1.82) for sample size 80 and R-4 were close to the theoretical mean and variance.

Overall, the Bartlett method had 23 of 60 pairs of means and variances close to their respective theoretical means and variances. The IPF method had 38 of 60 pairs of means and variances close to the theoretical means and variances. Both Goodman methods had 9 of 60 pairs of means and variances close to the theoretical means and variances. Of the methods, the IPF method was robust for most sampling distributions with sample sizes of 40 and above and was robust for two sampling distributions in the sample size 20 category with dimension 2 × 2 × 3. The Bartlett method was robust for sample sizes 60, 80, and 100 in the 2 × 2 × 3 dimension and sample size 100 in the 2 × 2 × 2 dimension. For all practical purposes, the Goodman method was not robust with respect to this secondary indicator of robustness.

Although it was informative to inspect the means and variances for the 60 combinations of dimension (3), sample size (5), and sampling distribution (4), it was more beneficial to investigate the means and variances in Table 10. Table 10 contains the means and variances of the chi-square statistics of the methods (Bartlett, IPF, Goodman 1, Goodman 2) after collapsing over sampling distribution. As in the discussion of Hypothesis 4, the

cross-classification of dimension and sample size can be of more practical use than the cross-classification which includes sampling distribution. The means and variances are considered close to the theoretical means and variances whenever both are approximately within 10% of the theoretical values.

For the 3 × 3 × 3 dimension, no Bartlett and Goodman means and variances satisfied the 10% criterion. The IPF means and variances for: sample size 60 (8.43, 17.21), sample size 80 (8.64, 16.76), and sample size 100 (8.60, 15.15) were within 10% of the theoretical mean, 8 and variance, 16.

For the 2 × 2 × 3 dimension, no Goodman means and variances were within 10% of the theoretical mean, 2 and variance, 4. The Bartlett means and variances for: sample size 60 (2.12, 4.00), sample size 80 (2.09, 3.97), and sample size 100 (2.10, 4.17) were close to the theoretical values. The IPF means and variances for sample size 40 (2.23, 4.25), sample size 60 (2.16, 4.36), sample size 80 (2.10, 4.05), and sample size 100 (2.11, 4.21) were close to the theoretical values.

For the 2 × 2 × 2 dimension, the Bartlett means and variances for sample size 40 (1.11, 2.02), sample size 60 (1.10, 2.11), sample size 80 (1.07, 2.07), and sample size 100 (1.04, 1.90) were close to the theoretical mean, 1 and variance, 2. The IPF means and variances for sample

size 40 (1.17, 2.21), sample size 60 (1.13, 2.29), sample size 80 (1.08, 2.15), and sample size 100 (1.04, 1.92) were close to the theoretical values. No Goodman means and variances satisfied the 10% criterion.

Scatter Diagrams

The scatter diagrams given in Appendices A, B, and C were used as a basis from which to conduct the linear regression analysis. Appendix A contains the scatter diagrams for the Bartlett chi-square statistics vs. the IPF chi-square statistics for the cross-classification of dimension by sample size. The only significant departures from linearity occurred for the 3 × 3 × 3 dimension. Differences between the Bartlett and IPF statistics occurred because the methods gave different expected values for the same contingency table. The different sets of expected values resulted in different values of the two statistics.

Example #1 in Table 23 is a 3 × 3 × 3 contingency table for which the Bartlett and IPF statistics were identical (20.54). The expected values are in parentheses below the corresponding observed values. It should be noted that, within rounding error, the sum of the cell deviations is zero.

Example #2 in Table 23 is a 3 × 3 × 3 contingency table for which the Bartlett (18.73) and the IPF (14.04) statistics were different. The Bartlett expected values are in parentheses below the corresponding observed

Table 23
Examples of 3 × 3 × 3 Contingency Tables

		Layer 1			Layer 2			Layer 3		
	Column	1	2	3	1	2	3	1	2	3
		Example #1[a]								
Row	1	0	3	4	6	2	3	3	1	2
		(2.76)	(1.40)	(2.84)	(3.96)	(3.04)	(4.00)	(2.27)	(1.56)	(2.17)
	2	4	2	1	4	1	5	3	4	2
		(3.13)	(1.52)	(2.35)	(4.05)	(2.97)	(2.98)	(3.82)	(2.51)	(2.66)
	3	5	1	5	0	6	2	3	3	5
		(3.11)	(3.09)	(4.81)	(1.91)	(2.99)	(3.02)	(2.91)	(3.92)	(4.17)
		a- Bartlett and IPF statistics = 20.54								
		Example #2[b]								
Row	1	1	1	1	4	4	4	1	6	6
		(0.46)	(1.09)	(1.45)	(1.77)	(4.28)	(4.95)	(3.76)	(5.63)	(3.61)
		(0.58)	(1.11)	(1.31)	(2.09)	(4.32)	(5.59)	(3.33)	(5.57)	(4.10)
	2	2	5	5	3	4	4	4	1	1
		(3.50)	(4.09)	(4.41)	(3.10)	(3.70)	(4.19)	(2.40)	(2.21)	(1.40)
		(3.63)	(4.08)	(4.29)	(3.07)	(3.69)	(4.24)	(2.30)	(2.24)	(1.47)
	3	4	2	5	0	2	6	2	2	0
		(3.04)	(2.82)	(5.14)	(2.12)	(2.02)	(3.86)	(0.84)	(1.17)	(2.00)
		(2.79)	(2.81)	(5.40)	(1.84)	(1.99)	(4.17)	(1.37)	(1.20)	(1.43)
		b- Bartlett statistic = 18.73 IPF statistic = 14.04								

values and the IPF expected values are below the corresponding Bartlett expected values. It should be noted that the sum of the cell deviations is zero (within rounding error).

Appendix B contains the scatter diagrams for Bartlett vs. Goodman 1 and IPF vs. Goodman 1. Appendix C contains the scatter diagrams for Bartlett vs. Goodman 2 and IPF vs. Goodman 2. As in Appendix A, the scatter diagrams were based upon the cross-classification of dimension by sample size. The scatter diagrams in Appendix B indicate more variability than the corresponding scatter diagrams in Appendix C (e.g. see Figure 28 in Appendix B and Figure 57 in Appendix C). This indicates that although the manner in which the ½s were added to the cell frequencies had little appreciable effect on the Goodman chi-square statistics (see the discussions of the results of the investigations of Hypotheses 1 to 6), enough variability was removed from the Goodman 2 statistics that there was a stronger linear relationship between Bartlett and Goodman 2 and IPF and Goodman 2 than there was between Bartlett and Goodman 1 and IPF and Goodman 1.

A number of scatter diagrams in Appendices B and C contained "outliers" (e.g. Figures 28 and 29 in Appendix B). Table 24 contains two 3 × 3 × 3 contingency tables taken from Figures 51 and 52. The Bartlett, IPF, Goodman 1, and Goodman 2 statistics were: 18.05, 18.05, 13.88, and 13.15 for Example #1 and were: 18.06, 18.06, 9.09, and 9.52 for

Table 24

Outliers for Goodman Statistics

		Layer 1			Layer 2			Layer 3		
	Column	1	2	3	1	2	3	1	2	3
		Example #1[a]								
Row	1	4	1	4	3	1	2	3	8	2
	2	2	0	3	1	2	3	5	1	3
	3	0	3	6	6	5	3	1	1	7
		a-Bartlett(IPF): 18.05, Goodman: 13.88, 13.15								
		Example #2[b]								
Row	1	0	3	3	1	1	4	3	6	2
	2	3	6	0	1	1	3	2	5	6
	3	5	4	0	4	0	6	3	1	7
		b-Bartlett(IPF): 18.06, Goodman: 9.09, 9.52								

Example #2. Example #2 had more empty cells than Example #1. Consequently, the number of empty cells in a given contingency table can affect the magnitude of the Goodman statistic regardless of the manner in which the ½s are added to the cell frequencies.

Linear Regression

IPF on Bartlett

The results of the linear regression analysis of the IPF chi-square statistics on the Bartlett chi-square statistics are reported in Tables 12 and 13. Table 12 contains the statistics related to the intercepts, slopes, and models for the cross-classification of the chi-square statistics by dimension and sample size. Table 13 contains the corresponding tests of normality of the residuals that resulted from the linear models specified in Table 12. The results given in Table 12 indicate that the models give almost perfect predictability from IPF to Bartlett for the 2 × 2 × 3 and 2 × 2 × 2 dimensions. All R^2s for these particular dimensions and sample sizes were 1. This indicates that the linear models accounted for 100% of the variability. For the 3 × 3 × 3 dimension, although the R^2s were not exactly 1 (.936, .901, .950, .991), they were quite high and indicate that the models in this particular dimension predicted very well the Bartlett statistics from the IPF statistics. All models in Table 12 were statistically significant at the .0001 level. The results of the tests of the normality of

residuals (Table 13) indicate that there is no reason to reject the hypotheses that the residuals represented random samples from normally distributed populations.

Goodman 1, 2: One Line

Tables 14, 15, and 16 contain the results of the linear regression analysis with predictors Goodman 1 and Goodman 2 chi-square statistics and dependent variables Bartlett and IPF chi-square statistics. The linear models in Table 14 represent those combinations of dimension and sample size which have corresponding scatter diagrams that indicated one linear relationship (see corresponding scatter diagrams in Appendices B and C). All models were statistically significant at the .0001 level (see Table 15). Although the R^2s were not as large as those for the IPF on Bartlett linear regression, only four were less than .700 (Goodman 1: 3 × 3 × 3 and sample size 20 (.436), 40 (.647), 60 (.672); Goodman 2: 3 × 3 × 3 and sample size 20 (.671). The comparison between models for Goodman 1 and Goodman 2 indicates that the Goodman 2 models had R^2s that were higher consistently than the corresponding R^2s for the Goodman 1 models. For example, for the 3 × 3 × 3 dimension and sample size 40 case, the Goodman 1 R^2s were .647 (IPF) and .810 (Bartlett) whereas the corresponding Goodman 2 R^2s were .828 (IPF) and .866 (Bartlett). This supports the conclusion that for the scatter diagrams, there was less variability in the Goodman 2 scatter diagrams than the Goodman 1 scatter

diagrams. In addition, for every combination of dimension and sample size in which the regression analysis was done on both the Bartlett and IPF statistics, the R^2s for the Bartlett regression were greater than the corresponding R^2s for the IPF regression. For example, for the 3 × 3 × 3 and sample size 40, the Bartlett R^2 was .810 and the IPF R^2 was .647 for Goodman 1. This indicates that the prediction from Goodman 1 or Goodman 2 was better for the Bartlett statistic than for the IPF statistic for certain combinations of dimension and sample size. It should be noted that in those cases where the Bartlett and IPF statistics agreed (see Table 12: 2 × 2 × 3 dimension and sample sizes 60, 80, 100 and 2 × 2 × 2 dimension and all sample sizes), it was not necessary to do the regression analysis for both the Bartlett and IPF statistics. Since the IPF method analyzed more tables than the Bartlett method, (see Table 5), the IPF statistic was chosen as the dependent variable in these particular cases. Finally, all tests for the normality of residuals (see Table 16) were statistically significant at the .01 level. This is to say that there is no reason to reject the hypotheses that the residuals represented random samples from normally distributed populations.

Goodman 1, 2: Two Lines

Tables 17, 18, and 19 contain the results of the linear regression analyses with predictors Goodman 1 and Goodman 2 chi-square statistics and the dependent variable

IPF chi-square statistic. The linear models in Table 17 represent those combinations of the 2 × 2 × 2 dimension and sample sizes 40, 60, 80, and 100 that had scatter diagrams which indicated two linear relationships (see corresponding scatter diagrams in Appendices B and C). Since the Bartlett and IPF statistics agreed so well for the 2 × 2 × 2 dimension (see Table 12), it was not necessary to do the regression analysis on Goodman 1 and Goodman 2 for both the Bartlett and IPF statistics. Since the IPF method analyzed more tables than the Bartlett method (see Table 5), the IPF statistic was used as the dependent variable.

All the models were statistically significant at the .0001 level (see Table 18) and all R^2s ranged from .904 (sample size 40 with at least one empty cell) to .994 (sample size 60 with no empty cells). The comparison between models for Goodman 1 and Goodman 2 indicates that the corresponding R^2s were not very different. The largest discrepancy occurred for sample size 40 with at least one empty cell (Goodman 1: .904, Goodman 2: .960). Finally, all tests for the normality of residuals (see Table 19) were statistically significant at the .01 level. This means that there is no reason to reject the hypotheses that the residuals represented random samples from normally distributed populations.

Goodman 1: Three Lines

Table 20 contains the results of the linear regression analysis with predictor Goodman 1 chi-square statistic and

dependent variable IPF chi-square statistic. The linear models in Table 20 represent those combinations of the 2 × 2 × 2 dimensions with sample sizes 20 and 40 which had scatter diagrams that indicated three linear relationships (see Figures 43 and 45 in Appendix B). Since the Bartlett and IPF statistics agreed so well for the 2 × 2 × 2 dimension (see Table 12), it was not necessary to do the regression analysis for both the Bartlett and IPF statistics. Since the IPF method analyzed more tables than the Bartlett method (see Table 5), the IPF statistic was used as the dependent variable.

All models in Table 20 were statistically significant at the .0001 level and all R^2s ranged from .784 (sample size 20 with at least two empty cells) to .995 (sample size 20 with no empty cells). Finally, all tests for normality of residuals (see Table 20) were statistically significant at the .01 level. This means that there is no reason to reject the hypotheses that the residuals represented random samples from normally distributed populations.

It should be noted that for the 2 × 2 × 2 dimension and sample size 40, the regression analysis was done twice (once on the assumption of only two lines and once on the assumption of three lines) because it was not very clear whether there were two lines or three lines in the scatter diagram (see Figure 45, Appendix B). The results for the three-line analysis indicate that the R^2 (.867) for more

than one empty cell was quite high and R^2 (.995) for no empty cells was slightly higher than the corresponding R^2 (.989) for the no empty cells classification under the two-line analysis. Therefore, the three-line analysis resulted in slightly better predictive models.

Unanalyzed Tables

Frequencies of Empty Cells

Table 21 contains the frequency distribution of the number of tables analyzed (A) and not analyzed (NA) of the Bartlett and IPF methods by dimension and sample size with respect to the number of empty cells in the tables. Frequencies were not reported for the Goodman method because the Goodman method analyzed all tables (see Table 2).

In general, for a given number of empty cells, the IPF method analyzed more tables than the Bartlett method. This was not surprising because of the flexibility of the IPF method with respect to empty cells (Fienberg, 1977).

In the 3 × 3 × 3 dimension (see Table 21) the break even point of the Bartlett method was three empty cells for sample sizes 40, 60, and 100 and two empty cells for sample size 80. In other words, for these number of empty cells, there was roughly a 50-50 chance that a table with three (or two) empty cells would be analyzed by the Bartlett method. For sample size 20, the break even point of the IPF method was 17 empty cells. For sample sizes 40 and 60, the break even point of the IPF method was three empty cells

and for sample sizes 80 and 100 it was two empty cells.

In the 2 × 2 × 3 dimension, the Bartlett method had difficulty in analyzing tables that had more than one empty cell. The break even point of the IPF method for sample sizes 20 and 40 was four empty cells. For sample sizes 60, 80, and 100, the IPF method had little, if any, difficulty in analyzing tables that had up to three empty cells.

In the 2 × 2 × 2 dimension, the break even point of the Bartlett method was one empty cell for all sample sizes. The IPF method had little, if any, difficulty in analyzing tables that had one empty cell.

These results indicate that the number of empty cells, the location of these empty cells, and the magnitudes of the nonempty cells are factors which determine whether or not the Bartlett or IPF methods can analyze a given table. This point is evident for the Bartlett method with the 2 × 2 × 2 dimension and one empty cell.

Negative Chi-Square Statistics

Since no negative chi-square statistics were reported (see Results Chapter), there should be some discussion of this point with respect to the Bartlett method. McNamee (1973) reported a number of negative Bartlett statistics. An inspection of his Bartlett algorithm for computing the Bartlett statistic uncovered two important points.

First, the series given in Equation 36 was allowed to accumulate until the sum reached the largest double

precision number that was representable by the computer. This number was used in the calculation of the Bartlett statistic. However, this was an error on McNamee's part because this artificial number is not the true value to be used in the calculation of the Bartlett statistic. Therefore, some of his Bartlett statistics were not necessarily correct. If the Bartlett statistic was negative, then McNamee interchanged columns with the hope of eliminating these negative statistics. In most cases, this procedure eliminated the negative statistics; but, again, the new positive statistics may be wrong because of the artificial number that could arise because of the nonconvergence of the series in Equation 36.

Second, even if the Bartlett statistic was positive before or after an interchange and the series given in Equation 36 did converge, the statistic may have been incorrect because of negative expected values. McNamee did not make provisions for the occurrence of negative expected values and so some of his positive Bartlett statistics may be wrong.

For example, consider the following 2 × 2 × 2 table:

1	1	0	0
1	8	8	1

The corresponding expected values were:

-.25	2.25	1.25	-1.25
2.25	6.75	6.75	2.25

The Bartlett statistic was -4.26. When the columns were interchanged within each layer (McNamee, 1973), the table became:

1	1	0	0
8	1	1	8

The corresponding expected values were:

3.60	-1.60	-2.60	2.60
5.40	3.60	3.60	5.40

The new Bartlett statistic was 3.91. Therefore, the Bartlett method gave an erroneous positive chi-square statistic.

For the present study, the Bartlett algorithm was written with safeguards for the nonconvergence of the series in Equation 36 and the negative expected values. If either of these particular points occurred during the evaluation of a given table, then the interchange procedure was used until a valid positive statistic occurred or the table was designated as "not analyzed" should the interchange procedure fail to yield a valid statistic.

Non Positive Degrees of Freedom

Table 22 contains a sample of the 4,865 tables that had zero or negative adjusted degrees of freedom (see Equation 101). If the degrees of freedom is zero or negative then this means that it is impossible to measure second order interaction by the IPF method (Fienberg, 1981). Therefore, these particular tables were designated as "not

analyzed" by the IPF method. The concepts of zero or negative degrees of freedom should be viewed in conjunction with the frequency distribution of the number of tables with a given number of empty cells (see Table 21). Although the number of cells that have zeros is important, the location of these zeros in the table is also important. For example, consider the following 2 × 2 × 2 tables:

Table A				Table B			
0	a	c	d	0	a	d	e
0	b	e	f	b	c	0	f

For Table A and Equation 101: $T_e=8$, $Z_e=2$, $T_p=7$, and $Z_p=1$. The adjusted degrees of freedom is 0. For Table B and Equation 101: $T_e=8$, $Z_e=0$, $T_p=7$, and $Z_p=0$. The adjusted degrees of freedom is 1. Therefore Table A cannot be measured for second order interaction and Table B can be measured for second order interaction.

Recapitulation of Results

Robustness

The Bartlett method was robust for 2 × 2 × 3 tables with sample sizes 40 to 100 and for 2 × 2 × 2 tables with sample sizes 20 to 100. In Table 5, the rejection rates for these particular combinations of dimension and sample size ranged from .043 to .056. From Equation 100 and the number of tables analyzed (see Table 5), the corresponding precisions ranged from .005 to .006. The precisions were based upon a hypothesized population parameter of .05. The

Bartlett univariate results for these particular combinations of dimension and sample size were, for the most part, supportive of the conclusion that the Bartlett method was robust.

The IPF method was not as robust as the Bartlett method and the IPF method was the least conservative method. This means that the IPF method had a tendency to reject H_0, no second order interaction, more often than either the Bartlett or Goodman methods. This was particularly true for: the $3 \times 3 \times 3$ dimension with sample sizes 20 to 80, the $2 \times 2 \times 3$ dimension with sample sizes 20 and 40, and the $2 \times 2 \times 2$ dimension with sample sizes 20 to 60. The rejection rates for these combinations of dimension and sample size ranged from .058 to .114 (see Table 5). The corresponding precisions (see Equation 100) ranged from .005 to .006 for a hypothesized population parameter of .05. For the remaining combinations of dimension and sample size ($3 \times 3 \times 3$ with sample size 100, $2 \times 2 \times 3$ with sample sizes 60 to 100, and $2 \times 2 \times 2$ with sample sizes 80 and 100), the IPF method was very robust. If it is important to protect against making a Type II error (falsely accepting H_0), then the IPF method should be used because of its tendency to have a high rejection rate. The IPF univariate results were not very supportive of the conclusion that the IPF method was robust for some of combinations of dimension and sample size. However, the discrepancies between the

univariate results and the rejection rates should be evaluated in terms of the portion of the theoretical chi-square distribution that is of interest. In the present study, the upper 5% of the χ^2 distribution was of interest. So while there may be discrepancies at the mean, these discrepancies probably are irrelevant in the upper 5% region of the χ^2 distribution.

The Goodman method was not very robust and was the most conservative method. This means that the Goodman method had a tendency not to reject H_0, no second order interaction. For all combinations of dimension and sample size, the Goodman rejection rates (Goodman 1 and Goodman 2) were less than .040 (see Table 5). Since the Goodman method analyzed every table, in Equation 100 with N=7,300, the precision was .005 for a hypothesized population parameter of .05. If it is important to protect against making a Type I error (falsely rejecting H_0), then the Goodman method should be used because of its tendency to have a low rejection rate. The univariate results of the Goodman method were supportive of the conclusions concerning the robustness of the Goodman method.

An additional consideration in the choice of methods to use for the testing of second order interaction is the number of empty cells in the table. For the 3 × 3 × 3 dimension, there is a reasonable chance that the Bartlett and IPF methods could be used when a given table has three empty

cells. For the 2 × 2 × 3 dimension, the Bartlett method worked well for one empty cell and the IPF method worked well for three empty cells. For the 2 × 2 × 2 dimension, the Bartlett and IPF methods worked well for one empty cell. The Goodman method analyzed all tables; but on the other hand, the Goodman method was very conservative.

Of the six hypotheses discussed in the present chapter, Hypothesis 4 (For a given sample size and dimension, the rejection rates are the same.) was the most important hypothesis. The results of the investigation of this particular hypothesis have more practical applications in contingency table analysis. A decision as to which method to use would be based in part upon the dimension and sample size.

Linear Regression

The results of the linear regression of the IPF statistic on the Bartlett statistic indicated that the only significant discrepancies between the two statistics occurred for the 3 × 3 × 3 dimension. However, even for this particular dimension, the linear models had extremely large R^2s. Therefore, if it were necessary to obtain an estimate of a Bartlett statistic from a given IPF statistic, the linear models given in Table 12 may be used.

If neither the Bartlett nor IPF methods can analyze a given contingency table, an estimate of these statistics may be found by computing the Goodman statistic and using

the appropriate model in Tables 14, 17, or 20. Although Goodman 2 had lower rejection rates than Goodman 1, the linear models for Goodman 2 had higher R^2s than the corresponding linear models for Goodman 1. Therefore, the Goodman statistic should be computed with a ½ added to every cell frequency.

An Application

Blau (1960) did an empirical study of social structures in terms of frequency distributions of the behavior of individuals or relationships among them. He wished to demonstrate that the structural effects of a social value can be isolated by showing that the association patterns of conduct are independent of whether or not an individual holds this value.

As a concrete example, Blau used the authoritarian disposition of an individual. Whether an individual possesses an authoritarian position or not, is this person more apt to discriminate against, say, minorities, if the person resides in a community where authoritarian values prevail than if he resides in a community where authoritarian values do not prevail?

Blau used data from a pilot study of a public assistance agency. In one section of this particular article, Blau investigated the effects of orientation toward work. In particular, case workers were put into two groups: case-work service (Layer 1) or checking eligibility for public

assistance (Layer 2). Within each layer, each case worker was classified as being individually oriented toward case work service (Column 1) or checking eligilibity (Column 2). Finally, individuals were further classified as reporting no conflict with auditors (Row 1) or some conflict with auditors (Row 2). These cross-classifications led to a 2 × 2 × 2 contingency table with sample size 60. No subject fell into more than one cell. Hence the classifications were independent. Following is this particular 2 × 2 × 2 contingency table.

	K=1		K=2	
	J=1	J=2	J=1	J=2
I=1	6	3	0	13
I=2	15	7	5	11

Blau analyzed the data in this particular contingency table by means of cell percentages and concluded that there was second order interaction. That is to say: The relationship between the individual's orientation (case work vs. eligibility) and another variable (no conflict with auditors vs. some conflict with auditors) is contingent on the prevalence of this value (orientation to case work vs. orientation to checking eligibility in the group).

The results of the analysis of this data by the Bartlett, IPF, and Goodman methods were as follows. The Bartlett and IPF statistics were identical (3.27). The Goodman 1 statistic was 2.02 and the Goodman 2 statistic was

1.87. When the appropriate linear model in Table 17 (2 × 2 × 2, sample size 60, at least one empty cell) was used, the predicted Bartlett (or IPF) statistic was 3.82 for Goodman 1 and was 3.21 for Goodman 2. Therefore, based upon these results, the hypothesis of no second order interaction would not be rejected. This contradicts Blau's conclusion and indicates that further investigations should be conducted in this particular area.

Recommendations for Further Study

The results of the present study raised some provocative questions. A number of empty cells in the contingency table presents a problem when the Bartlett method is used. The procedure adopted for this present study was to set first estimates of cell deviations to ½ wherever zeros appeared in the table. This was a purely computational technique used to get the iterative procedure started. However, this procedure did not guarantee the existence of a solution. Perhaps the number of tables analyzed by the Bartlett method could be increased if some other starting point for first estimates was used.

A second consideration related to the use of the Bartlett method is: Is there a relationship among the cells in terms of the number of empty cells, minimum nonzero cell frequencies, and the relative location of these zeros and minimum values?

A third consideration when the Bartlett method is

used is the interchange procedure used by McNamee (1973) in order to eliminate the negative chi-square values. Suppose every possible interchange is performed on those tables which had different Bartlett and IPF statistics. Under what, if any, conditions will a Bartlett solution match an IPF solution?

A fourth consideration when the Bartlett method is used is the solution to the system of cubic equations. When a solution occurs which yields negative expected values, how should this solution be interpreted? In other words, in the simplest case, a cubic equation can have one real or three real roots. Therefore, when this solution occurs (i.e. one that yields negative expected values), under what conditions can the appropriate solution be found?

With respect to the IPF method, are there contingency tables that have zero or negative degrees of freedom; but yet the second order interaction can be measured? Fienberg (1981) claims that no such example has been found.

With respect to the Goodman method, there was less variability in the Goodman statistics when a ½ was added to every cell than when a ½ was added to only the empty cells. The use of ½ is the recommended procedure for large sample theory. However, for small samples, a different constant may be appropriate.

The linear regression analysis was done as a secondary exploratory study. Is it appropriate to use these models

to estimate the Bartlett or IPF statistics?

Therefore, the present study uncovered some interesting results about the robustness of the three methods (Bartlett, IPF, Goodman). At the same time, questions were raised that require further investigations. The answers to these and other questions will undoubtedly lead to improvements on the present study.

SUMMARY

The overall purpose of the present study was to investigate the robustness of three methods (Bartlett, Iterative Proportional Fitting, Goodman) for measuring second order interaction in three dimensional contingency tables when the sample sizes were small. A Monte Carlo technique generated 3 × 3 × 3, 2 × 2 × 3, and 2 × 2 × 2 contingency tables for the following sample sizes: 20, 40, 60, 80, 100. In addition, four multinomial distributions for each dimension were used as underlying sampling distributions. Each sampling distribution was used to generate 1,825 contingency tables for each sample size. Consequently 109,500 contingency tables were generated for the 60 combinations of dimension (3), sample size (5), and sampling distribution (4).

Each table was tested for second order interaction by means of four methods: Bartlett, Iterative Proportional Fitting (IPF), Goodman 1, and Goodman 2. For Goodman 1, a ½ was added to each empty cell and for Goodman 2, a ½ was added to every cell before the table was tested for second order interaction by the Goodman method.

The primary indicator of robustness was the rejection rate of each method for various combinations of dimension, sample size, and sampling distribution. In particular the

cross-classificaion of dimension and sample size gave the most practical results. The Bartlett method was robust for 2 × 2 × 3 tables with sample sizes 40 to 100 and was robust for all sample sizes in the 2 × 2 × 2 dimension. The IPF method was robust for: the 3 × 3 × 3 dimension with sample size 100, the 2 × 2 × 3 dimension with sample sizes 60, 80, and 100, and the 2 × 2 × 2 dimension with sample sizes 80 and 100. In these particular cases, the empirical rejection rates were within .01 of the theoretical rejection rate, .05. The Goodman method was not robust and consistently had rejection rates less than .04.

For exploratory purposes, linear regression analysis was done on the chi-square statistics of the four methods. The Goodman 1 and Goodman 2 statistics were predictors of the Bartlett and IPF statistics and the IPF statistic was the predictor of the Bartlett statistic. All linear models were statistically significant at the .0001 level and there were only nine R^2s (out of a possible 69 linear models) less than .800. The models that involved the Goodman statistic provided a means of estimating the Bartlett or IPF statistics when either of these latter two methods was unable to analyze a given contingency table.

REFERENCES

Amick, D. J., & Walberg, H. J. Introductory multivariate analysis: For educational, psychological, and social research. Berkeley, CA.: McCutchan, 1975.

Anderson, R. L., & Bancroft, T. A. Statistical theory in research. New York: McGraw-Hill, 1952.

Bartlett, M. S. Contingency table interactions. Journal of the Royal Statistical Society, 1935, Supplement 2, 248-252.

Berkson, M. D. Application of the logistic function to bio-assay. Journal of the American Statistical Society 1944, 39, 357-365.

Berkson. J., & Nagnur, B. N. A note on the minimum χ^2 estimate, and a L.A.M.S.T. χ^2 in the "no interaction" problem. Journal of the American Statistical Association, 1974, 69, 1038-1040.

Bhapkar, V. P. A note on the equivalence of two test criteria for hypotheses in categorical data. Journal of the American Statistical Association, 1966, 61, 228-235.

Bhapkar, V. P., & Koch, G. G. On the hypotheses of "no interaction" in contingency tables. Biometrics, 1968, 24, 567-594. (a)

Bhapkar, V. P., & Koch, G. G. Hypotheses of "no interaction" in multidimensional contingency tables. Technometrics, 1968, 10, 107-123. (b)

Bhat, B. R., & Kulkarni, S. R. Lamp tests of linear and loglinear hypotheses in multinomial experiments. Journal of the American Statistical Association, 1966, 61, 236-245.

Birch, M. W. Maximum likelihood in three-way contingency tables. Journal of the Royal Statistical Society, 1963, 25B, 220-233.

Bishop, Y. M. M. Full contingency tables, logits, and split contingency tables. Biometrics, 1969, 25, 383-400.

Bishop, Y. M. M., Fienberg, S. E., & Holland, P. W. Discrete multivariate analysis theory and practice. Cambridge: MIT Press, 1975.

Blau, P. M. Structural effects. American Sociological Review. 1960, 25, 178-193.

Cochran, W. G. Methods for strengthening χ^2. Biometrics, 1954, 10, 417-451.

Cox, M. A. A., & Plackett, R. L. Small samples in contingency tables Biometrika, 1980, 67, 1-13.

Craddock, J. M. Testing the significance of a 3 x 3 contingency table. The Statistician, 1966, 16, 87-94.

Craddock, J. M., & Flood, C. R. The distribution of the χ^2 statistic in small contingency tables. Applied Statistics, 1970, 19, 173-181.

Darroch, J. N. Interactions in multi-factor contingency tables. Journal of the Royal Statistical Society, 1962, 24B, 251-263.

Darroch, J. N., & Ratcliff, D. Generalized iterative scaling for log-linear models. The Annals of Mathematical Statistics, 1972, 43, 1470-1480.

David, F. N. Two combinatorial tests of whether a sample has come from a given population. Biometrika, 1950, 37, 97-110.

Deming, W. E., & Stephan, F. F. On a least squares adjustment of a sampled frequency table when the expected marginal totals are known. Annals of Mathematical Statistics, 1940, 11, 427-444.

Edwards, A. L. On "the use and misuse of the chi-square test" - The case of the 2 x 2 contingency table. Psychological Bulletin, 1950, 47, 341-346.

Fienberg, S. E. The analysis of cross-classified categorical data. Cambridge: MIT Press, 1977.

Fienberg, S. E., & Holland, P. W. Simultaneous estimation of multinomial cell probabilities. Journal of the American Statistical Association, 1973, 68, 683-691.

Forsythe, G. E., & Moler, C. B. Computer solution of linear algebraic systems. Englewood Cliffs, N.J.: Prentice-Hall, 1967.

Freeman, G. H., & Halton, J. H. Note on an exact treatment of contingency, goodness of fit and other problems of significance. Biometrika, 1951, 38, 141-149.

Gart, J. J. Interaction tests for 2 × s × t contingency tables. Biometrika, 1972, 59, 309-317.

Gart, J. J., & Zweifel, J. R. On the bias of various estimators of the logit and its variance with application to quantal bioassay. Biometrika, 1967, 54, 181-187.

Gokhale, D. V. An iterative procedure for analysing log-linear models. Biometrics, 1971, 27, 681-687.

Gokhale, D. V., & Kullback, S. The information in contingency tables. New York: Marcel Dekker, 1978.

Good, I. J. Maximum enthopy for hypothesis formulation, especially for multidimensional contingency tables. Annals of Mathematical Statistics, 1963, 34, 911-934.

Good, I. J., Gover, T. N., & Mitchell, G. J. Exact distributions for χ^2 and for the likelihood-ratio statistic for the equiprobable multinomial distribution. Journal of the American Statistical Association, 1970, 65, 267-283.

Goodman, L. A. Measures of association for cross-classification II: Further discussion and references. Journal of the American Statistical Association, 1959, 54, 132-137.

Goodman, L. A. On Plackett's test for contingency table interaction. Journal of the Royal Statistical Society, 1963, 25B, 179-188.

Goodman, L. A. Interactions in multidimensional contingency tables. Annals of Mathematical Statistics, 1964, 35, 632-646. (a)

Goodman, L. A. Simple methods for analyzing three-factor interaction in contingency tables. Journal of the American Statistical Association, 1964. 59, 319-352. (b)

Goodman, L. A. The multivariate analysis of qualitative data: Interactions among multiple classifications. Journal of the American Statistical Association, 1970, 65, 226-256.

Goodman, L. A. Partitioning of chi-square, analysis of marginal contingency tables, and estimation of expected frequencies in multidimensional contingency tables. Journal of the American Statistical Association, 1971, 66, 339-344.

Goodman, L. A. On the relationship between two statistics pertaining to tests of three-factor interaction in contingency tables. Journal of the American Statistical Association, 1975, 70, 624-625.

Goodman, L. A., & Kruskal, W. H. Measures of association for cross classifications. Journal of the American Statistical Association, 1954, 49, 732-764.

Grizzle, J. E., Starmer, C. F., & Koch, G. G. Analysis of categorical data by linear models. Biometrics, 1969, 25, 489-504.

Grizzle, J. E., & Williams, O. D. Log linear models and tests of independence for contingency tables. Biometrics, 1972, 28, 137-156.

Haberman, S. J. The analysis of frequency data. Chicago: University of Chicago Press, 1974.

Haberman, S. J. Analysis of qualitative data (Vol. 1). New York: Academic Press, 1978.

Haberman, S. J. Analysis of qualitative data (Vol. 2). New York: Academic Press, 1979.

Haldane, J. B. S. The estimation and significance of the logarithm of a ratio of frequencies. Annals of Human Genetics, 1955, 20, 309-311.

Hays, W. L. Statistics for the social sciences (2nd ed.). New York: Holt, Rinehart, and Winston, 1973.

Ireland, C. T., & Kullback, S. Contingency tables with given marginals. Biometrika, 1968, 55, 179-188.

Kastenbaum, M. A. A note on the additive partitioning of chi-square in contingency tables. Biometrics, 1960, 16, 416-421.

Kastenbaum, M. A., & Lamphear, D. E. Calculation of chi-square to test the no three-factor interaction hypothesis. Biometrics, 1959, 15, 107-115.

Kavanagh, J. A. A Monte Carlo study of the polynomial discriminant method for pattern recognition. Unpublished doctoral dissertation, University of Minnesota, 1972.

Korducki, L. G. Interaction in three-dimensional contingency tables with small cell sizes: A limited Monte Carlo study. Unpublished master of science thesis, University of Chicago, 1977.

Krishnaswami, P., & Nath, R. Bias in multinomial classification. Journal of the American Statistical Association, 1968, 63, 298-303.

Ku, H. H., & Kullback, S. Iteraction in multidimensional contingency tables: An informational theoretic approach. Journal of Research of the National Bureau of Standards, 1968, 72B, 159-199.

Ku, H. H., & Kullback, S. Loglinear models in contingency table analysis. The American Statistician, 1974, 28, 115-122.

Ku, H. H., Varner, R. N., & Kullback, S. On the analysis of multidimensional contingency tables. Journal of the American Statistical Association, 1971, 66, 55-64.

Kullback, S., Kupperman, M., & Ku, H. H. Tests for contingency tables and Markov chains. Technometrics, 1962, 4, 573-585.

Lancaster, H. O. The derivation and partition of χ^2 in certain discrete distributions. Biometrika, 1949, 36, 117-129.

Lancaster, H. O. Complex contingency tables treated by the partition of χ^2. Journal of the Royal Statistical Society, 1951, 13B, 242-249.

Larntz, K. Small-sample comparisons of exact levels for chi-squared goodness-of-fit statistics. Journal of the American Statistical Association, 1978, 73, 253-263.

Lewis, B. N. On the analysis of interaction in multidimensional contingency tables. Journal of the Royal Statistical Society, 1962, 125A, 88-117.

Lewis, D., & Burke, C. J. The use and misuse of the chi-square test. Psychological Bulletin, 1949, 46, 433-489.

Lewis, D., & Burke, C, J. Further discussion of the use and misuse of the chi-square test. Psychological Bulletin, 1950, 47, 347-355.

Lewontin, R. C., & Felsenstein, J. The robustness of homogeneity tests in 2 × n tables. Biometrics, 1965, 21, 19-33.

Lindley, D. V. The Bayesian analysis of contingency tables. Annals of Mathematical Statistics, 1964, 35, 1622-1643.

Margolin, B. H., & Light, R. J. An analysis of variance for categorical data, II: Small sample comparisons with chi-square and other competitors. Journal of the Statistical Association, 1974, 69, 755-764.

Marsaglia, G., Ananthanarayanan, K., & Paul, N. How to use the McGill random number package "Super-Duper". Montreal: McGill University, 1972.

Mayo, S. T. Interactions among categorical variables. Educational and Psychological Measurement, 1961, 21, 839-858.

McNamee, R. J. Robustness of homogeneity tests in parallelepiped contingency tables. Unpublished doctoral dissertation, Loyola University of Chicago, 1973.

Miller, A. J. A Monte Carlo study of Pearson and log-linear chi-square one sample tests with small n. Unpublished doctoral dissertation, Loyola University of Chicago, 1979.

Mood, A. Introduction to the theory of statistics (1st ed.). New York: McGraw-Hill, 1950.

Mosteller, F. Association and estimation in contingency tables. Journal of the American Statistical Association, 1968, 63, 1-28.

Nagnur, B. N. L.A.M.S.T. and the hypotheses of no three factor interaction in contingency tables. Journal of the American Statistical Association, 1969, 64, 207-215.

Nass, C. A. G. The χ^2 test for small expectations in contingency tables, with special reference to accidents and absenteeism. Biometrika, 1959, 46, 365-385.

Norton, H. W. Calculation of chi-square for complex contingency tables. Journal of the American Statistical Association, 1945, 40, 251-258.

Odoroff, C. L. A comparison of minimum logit chi-square estimation and maximum likelihood estimation in 2 × 2 × 2 and 3 × 2 × 2 contingency tables: Tests for interaction. Journal of the American Statistical Association, 1970, 65, 1617-1631.

Patil, K. D. Interaction test for three-dimensional contingency tables. Journal of the American Statistical Association, 1974, 69 164-168.

Plackett, R. L. A note on interactions in contingency tables. Journal of the Royal Statistical Society, 1962, 24B, 162-166.

Reynolds, H. T. The analysis of cross-classifications. New York: Free Press (Macmillan), 1977.

Roscoe, J. T., & Byars, J. A. An investigation of the restraints with respect to sample size commonly imposed on the use of the chi-square statistic. Journal of the American Statistical Association, 1971, 66, 755-759.

Roy, S. N., & Kastenbaum, M. A. On the hypothesis of "no interaction" in a multi-way contingency table. Annals of Mathematical Statistics, 1956, 27, 749-757.

Shaffer, J. P. Multidimensional contingency tables. Psychological Bulletin, 1971, 79, 127-141.

Shanawany, M. R. An illustration of the accuracy of the χ^2 approximation. Biometrika, 1936, 28, 179-187.

Shapiro, S. S., & Wilk, M. B. An analysis of variance test for normality (complete samples). Biometrika, 1965, 52, 591-611.

Simpson, E. H. Interpretation of interaction in contingency tables, Journal of the Royal Statistical Society, 1951, 13B, 238-241.

Slakter, M. J. Comparative validity of the chi-square and two modified chi-square goodness-of-fit tests for small but equal expected frequencies. Biometrika, 1966, 53, 619-622.

Slakter, M. J. Accuracy of an approximation to the power of the chi-square goodness-of-fit test with small but equal expected frequencies. Journal of the American Statistical Association, 1968, 63, 912-924.

Snedecor, G. W. Chi-square of Bartlett, Mood, and Lancaster in a 2^3 contingency table. Biometrics, 1958, 14, 560-562.

Snedecor, G. W., & Cochran, W. G. Statistical Methods (6th ed.). Ames, Iowa: Iowa State University Press, 1967.

Statistical Analysis System, Inc. SAS user's guide, 1979 edition. Raleigh, N.C.: Author, 1979.

Stephens, M. A. EDF statistics for goodness of fit and some comparisons. Journal of the American Statistical Association, 1974, 69, 730-736.

Stuart, A. The estimation and comparison of strengths of association in contingency tables. Biometrika, 1953, 40, 105-110.

Tate, M. W., & Hyer, L. A. Significance values for an exact multinomial test and accuracy of the chi-square approximation (Final report). Washington, D.C.: Bureau of Research, Office of Education, 1969.

Tate, M. W., & Hyer, L. A. Inaccuracy of the χ^2 test of goodness of fit when expected frequencies are small. Journal of the American Statistical Association, 1973, 68, 836-841.

Wise, M. E. Multinomial probabilities and the χ^2 and X^2 distributions. Biometrika, 1963, 50, 145-154.

Woolf, B. On estimating the relation between blood group and disease. Annals of Human Genetics, 1955, 19, 251-253.

Zahn, D. A., & Roberts, G. C. Exact χ^2 criterion tables with cell expectations one: An application to Coleman's measure of consensus. Journal of the American Statistical Association, 1971, 66, 145-148.

Zelen, M. The analysis of several 2×2 contingency tables. Biometrika, 1971, 58, 129-137.

REFERENCE NOTES

Fienberg, S. E. Personal communication, January 26, 1981.

Haberman, S. J. Personal communication, September 4, 1980.

Haberman, S. J. Personal communication, January 22, 1981.

APPENDIX A

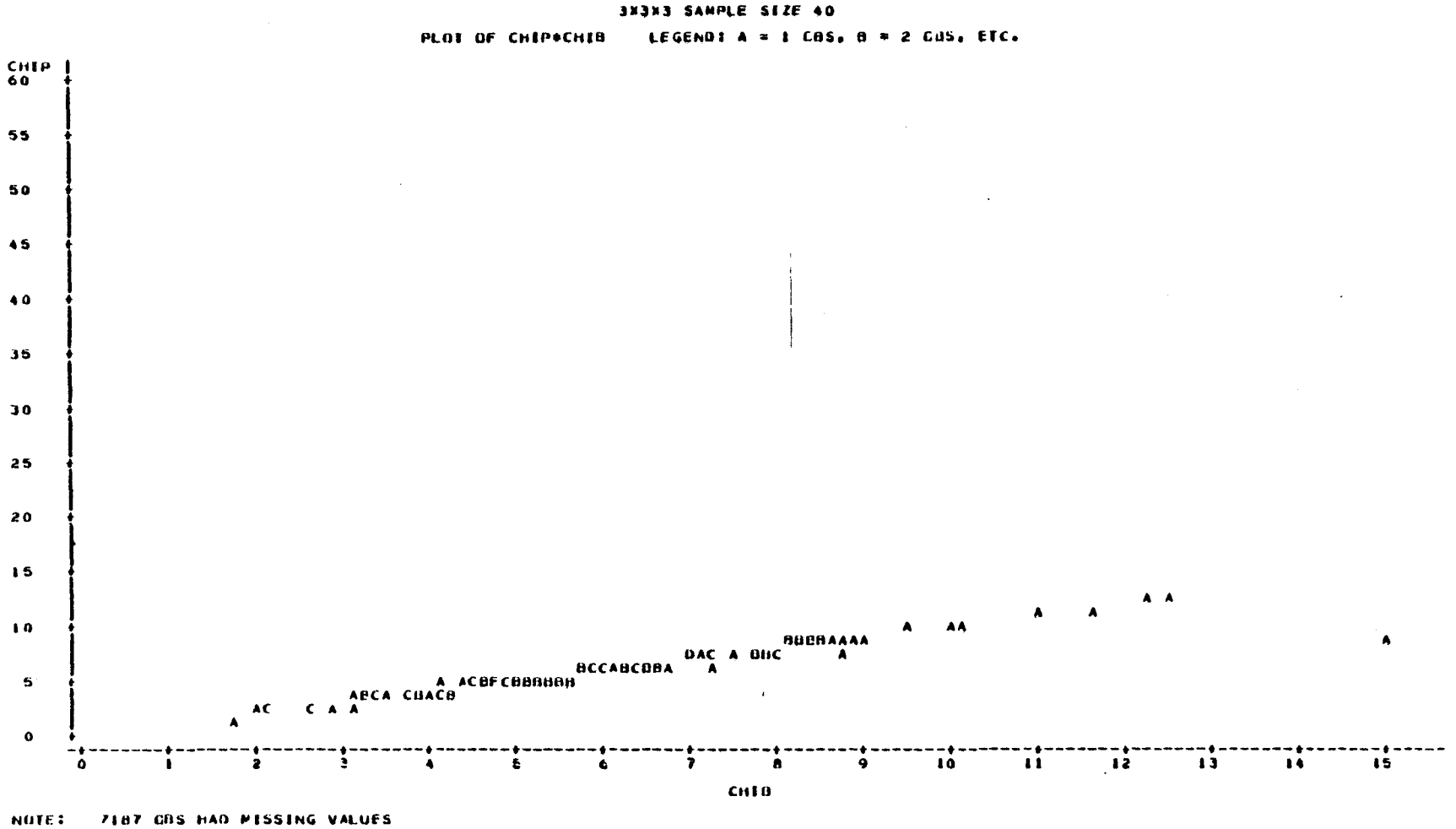
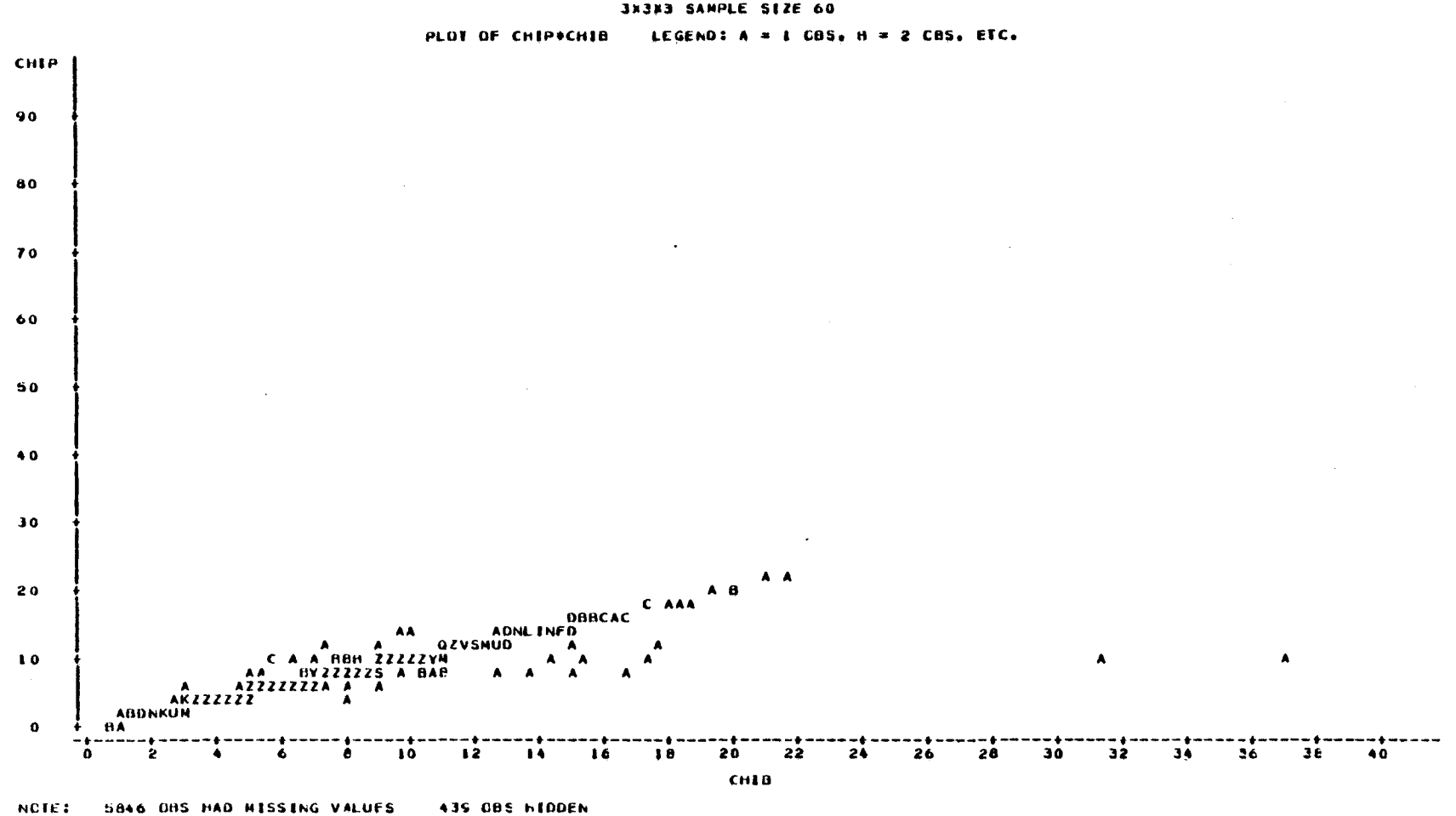


Figure 9. Scatter Diagram indicating one line: Bartlett vs. IPF



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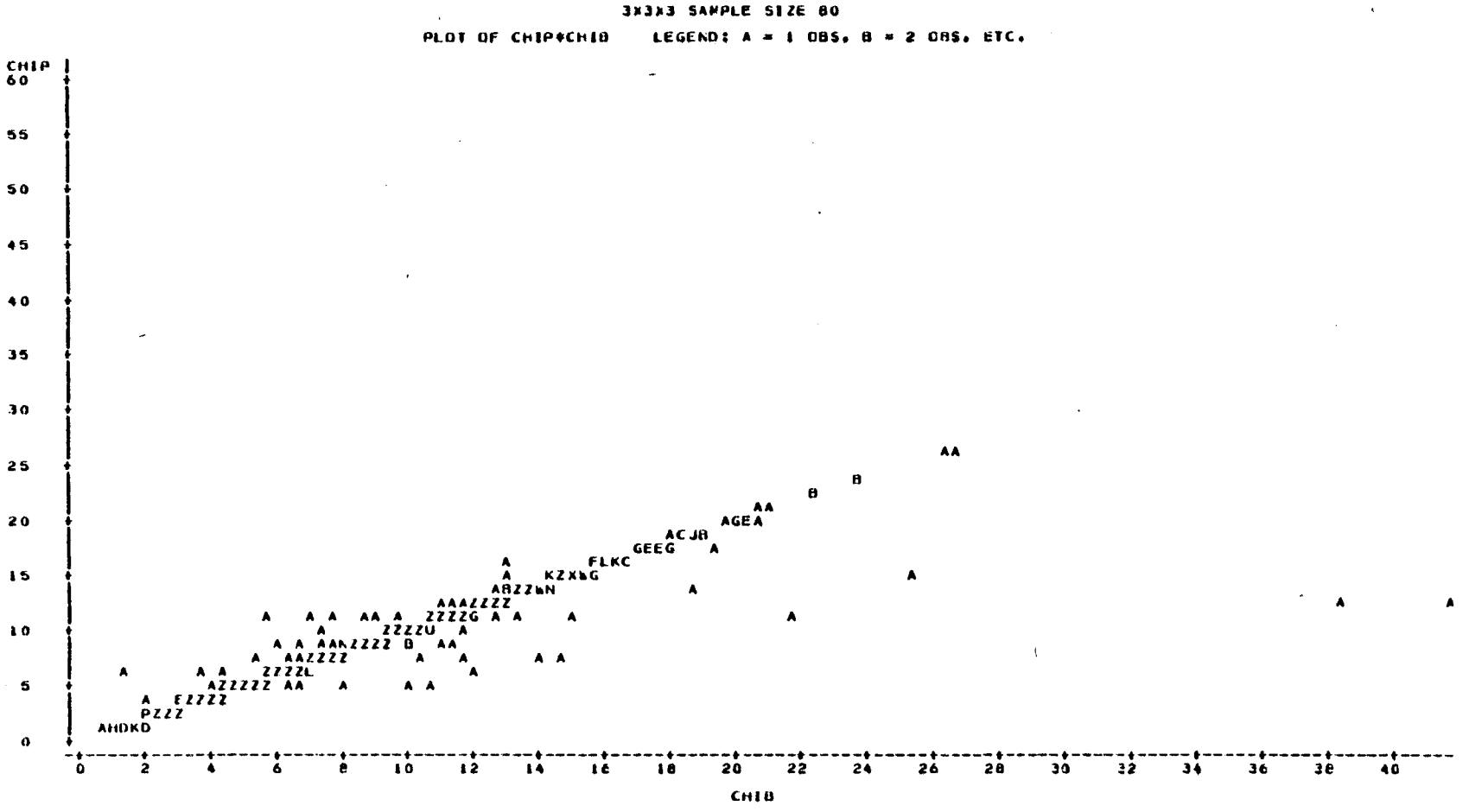

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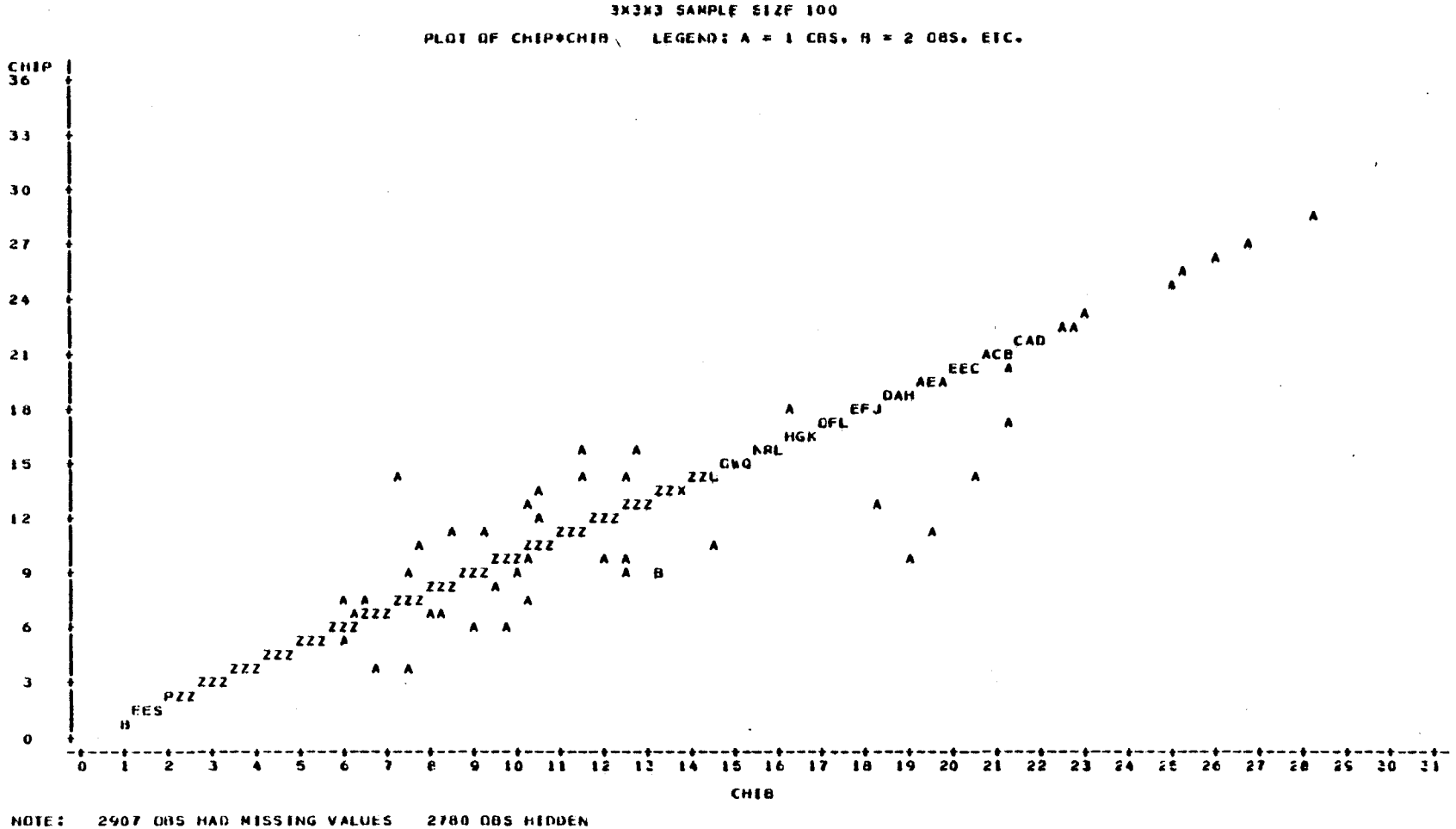

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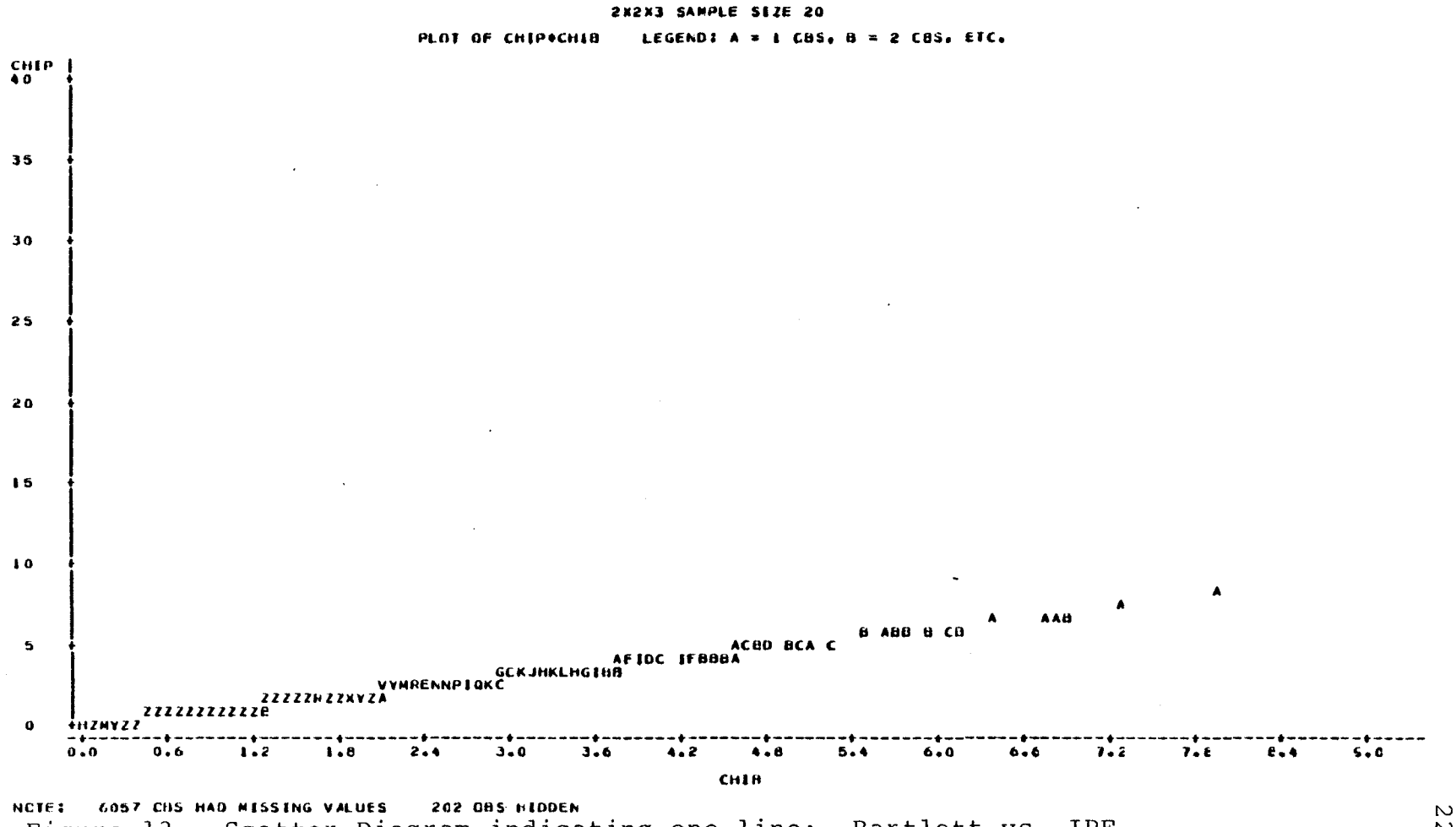

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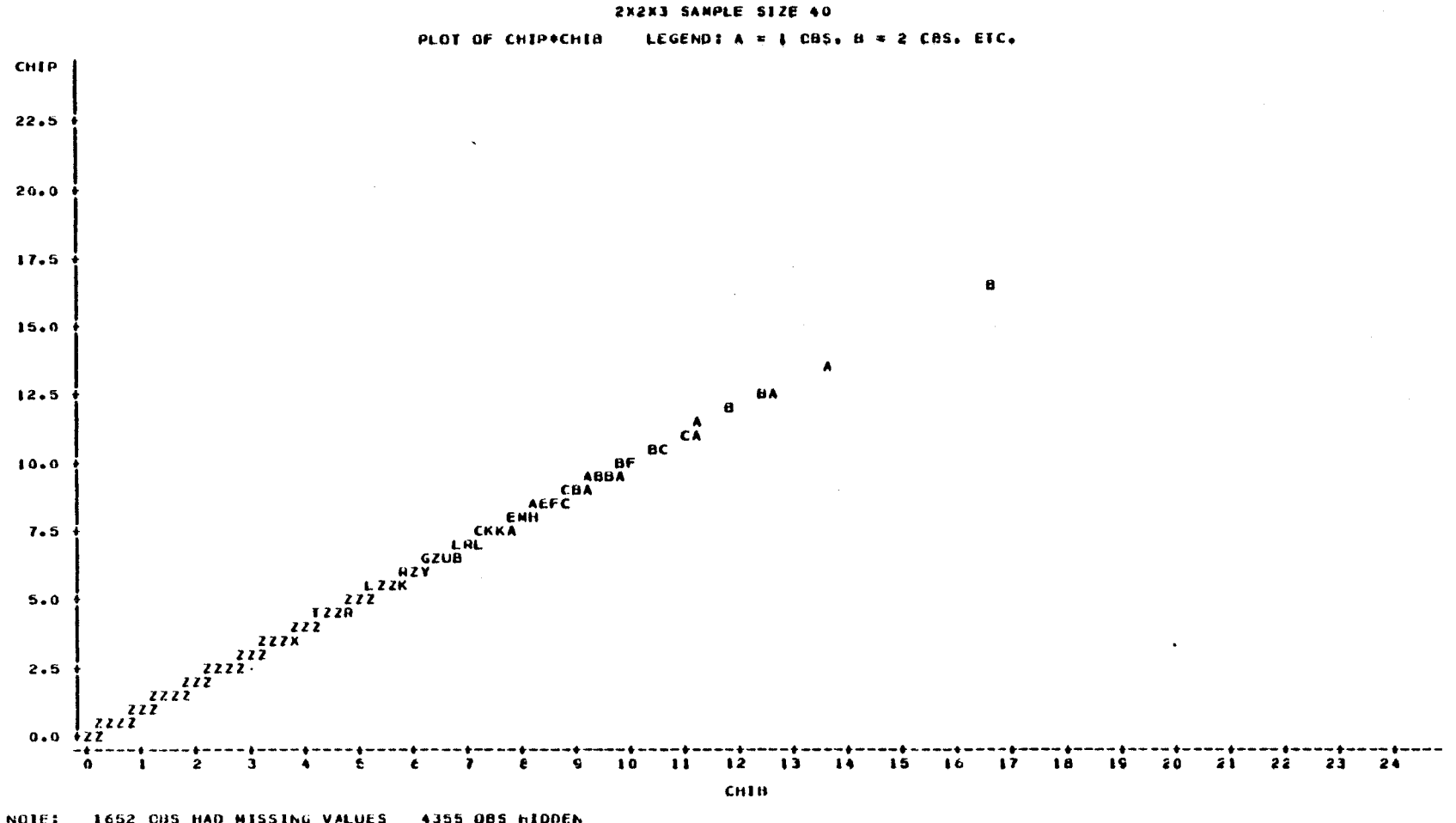

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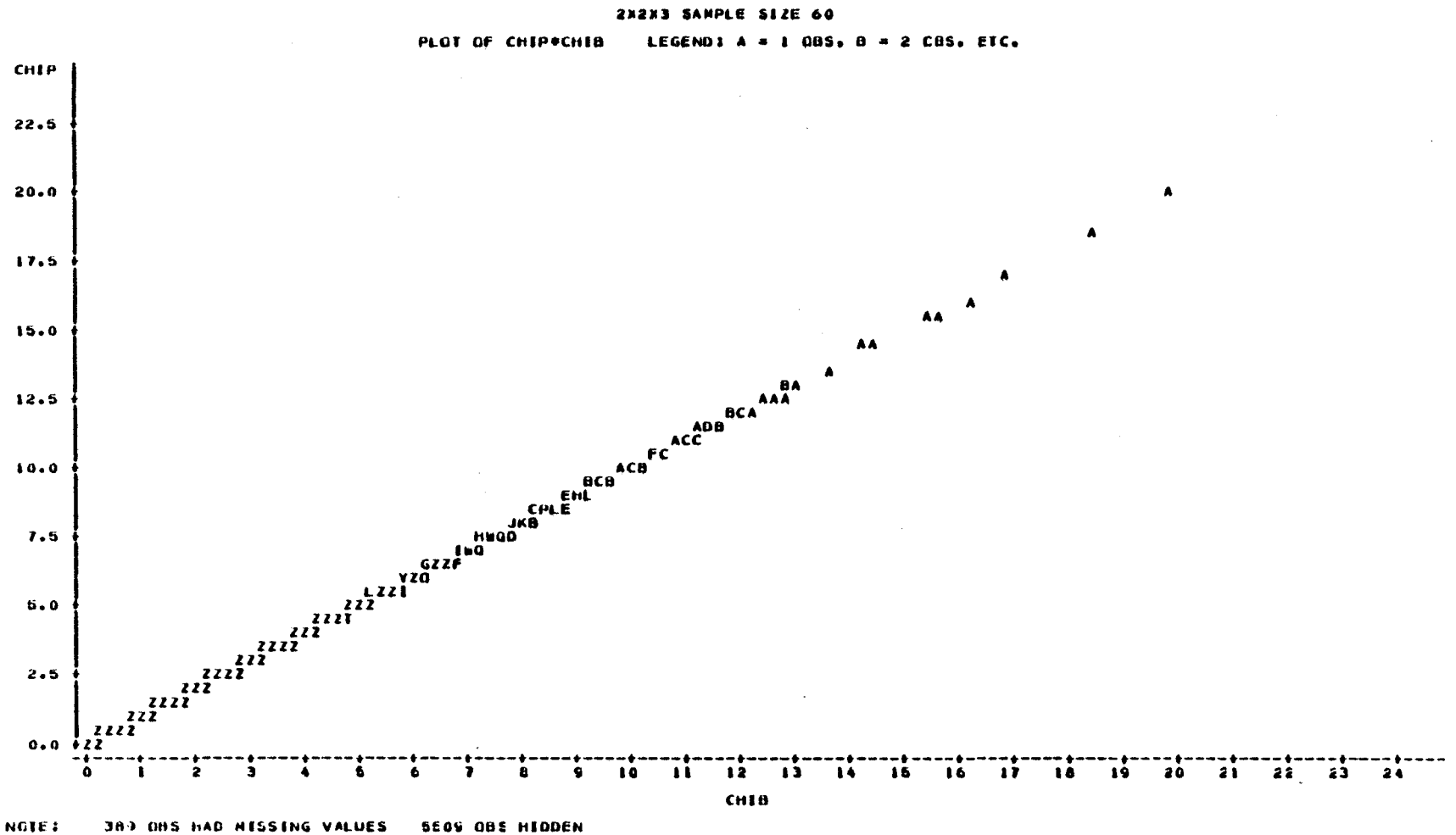

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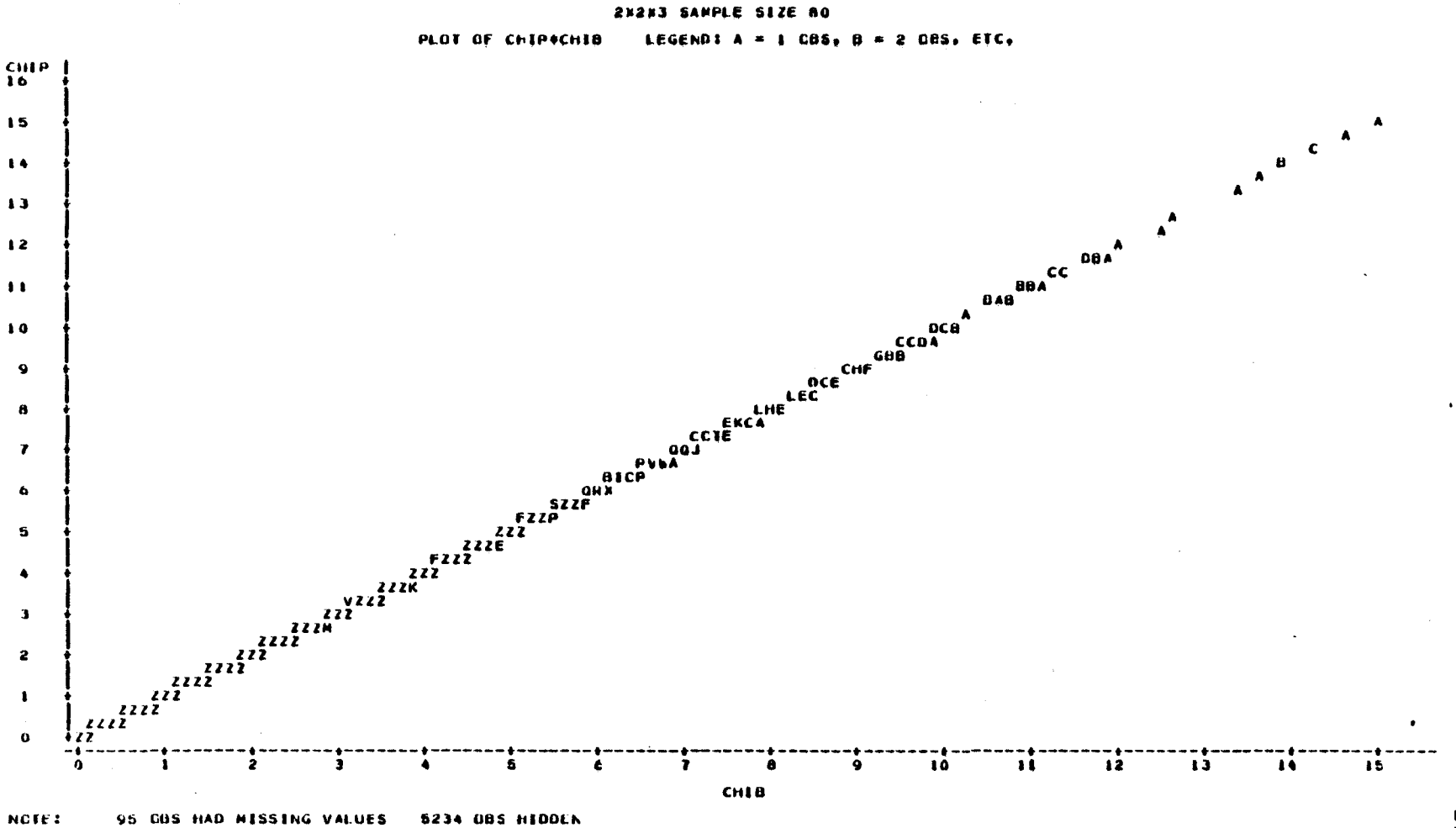

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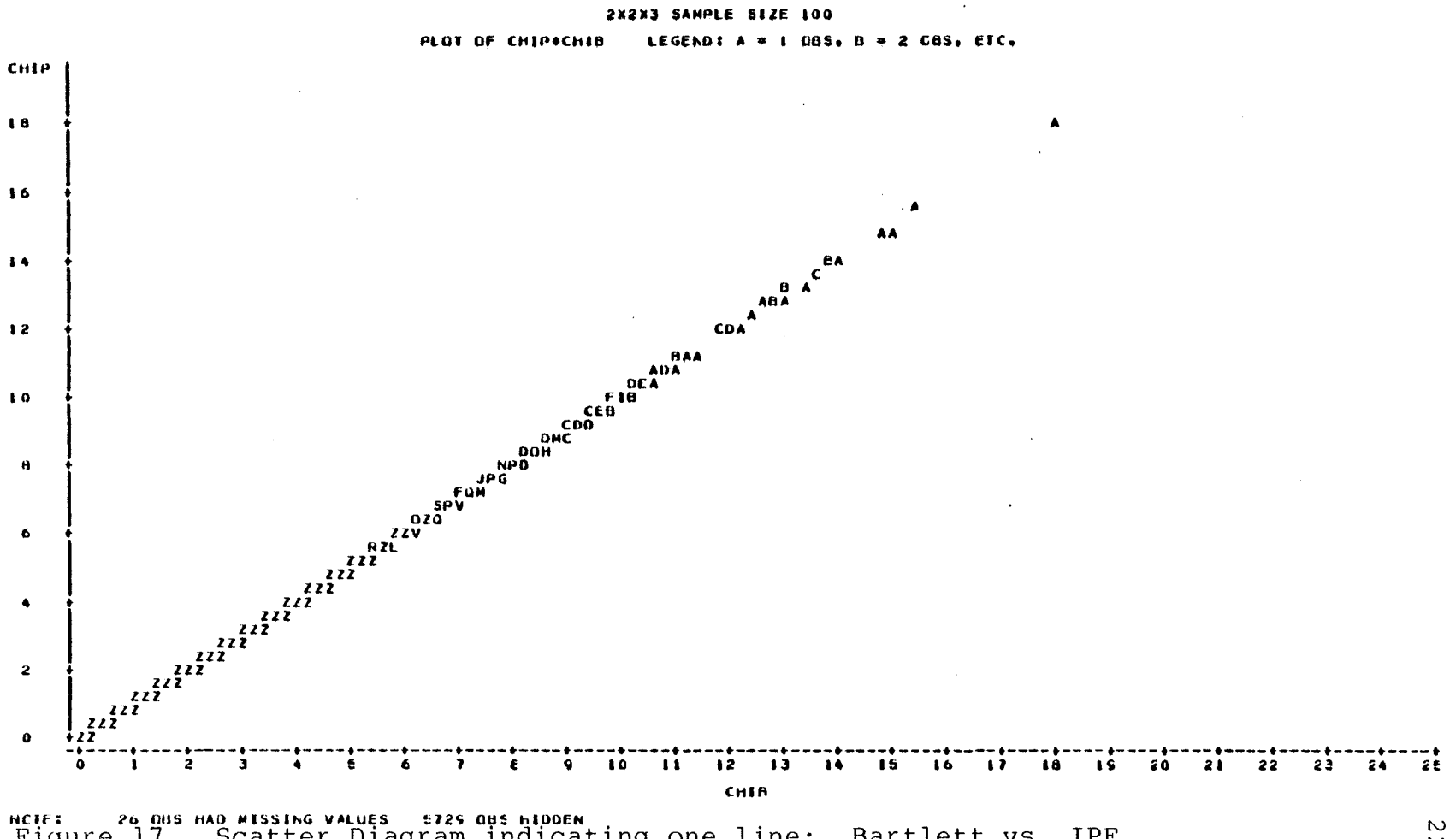

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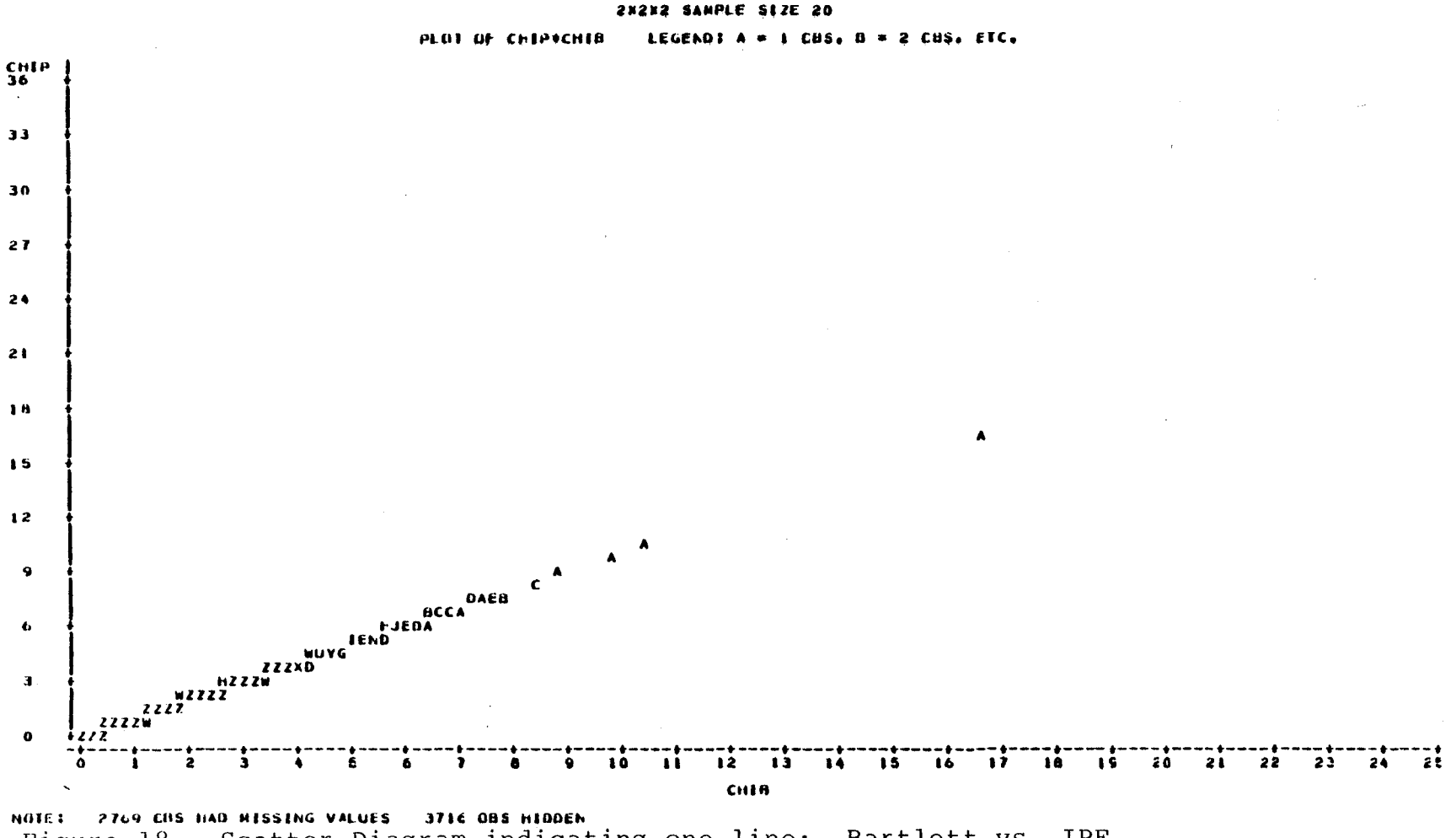

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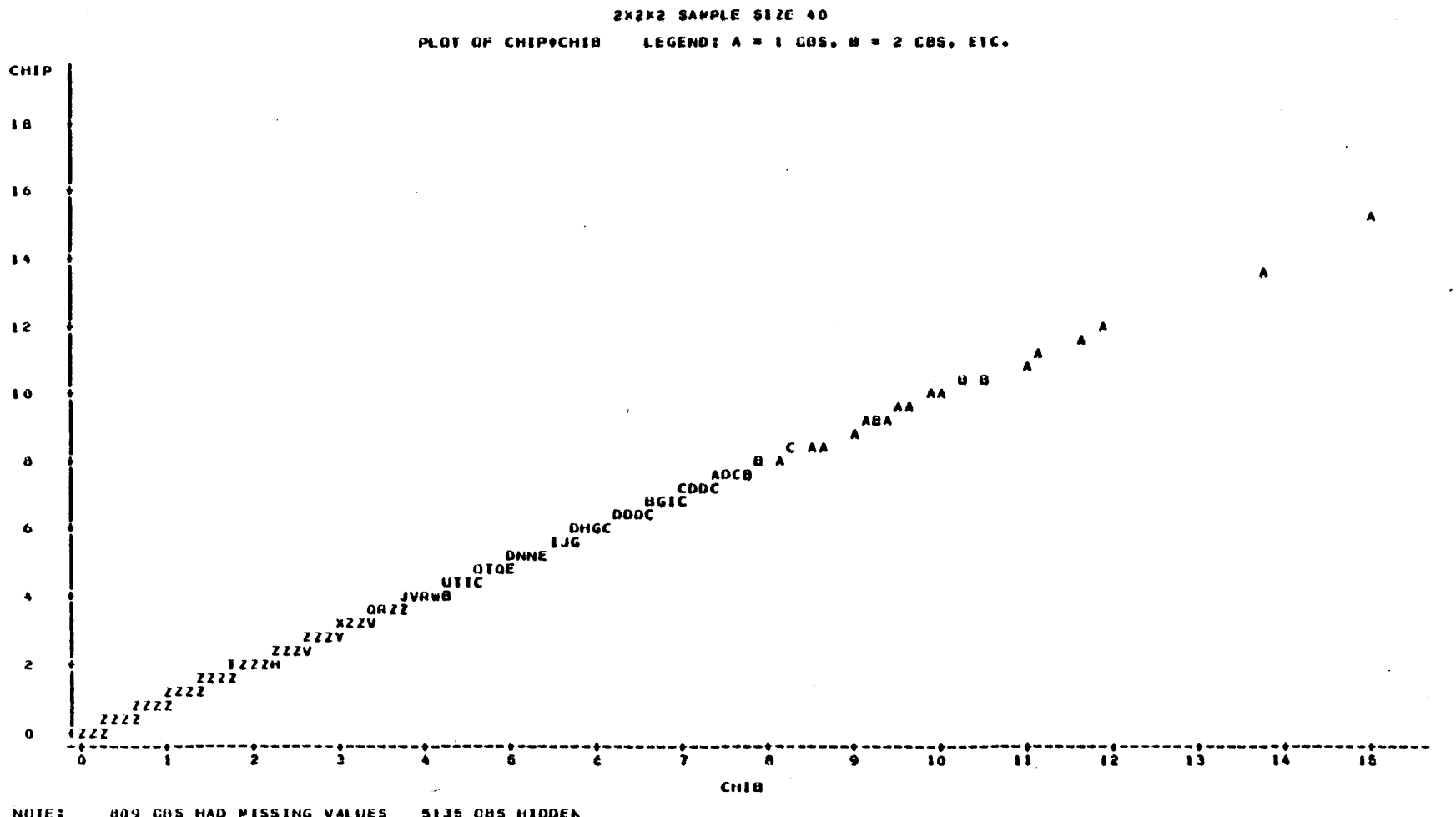

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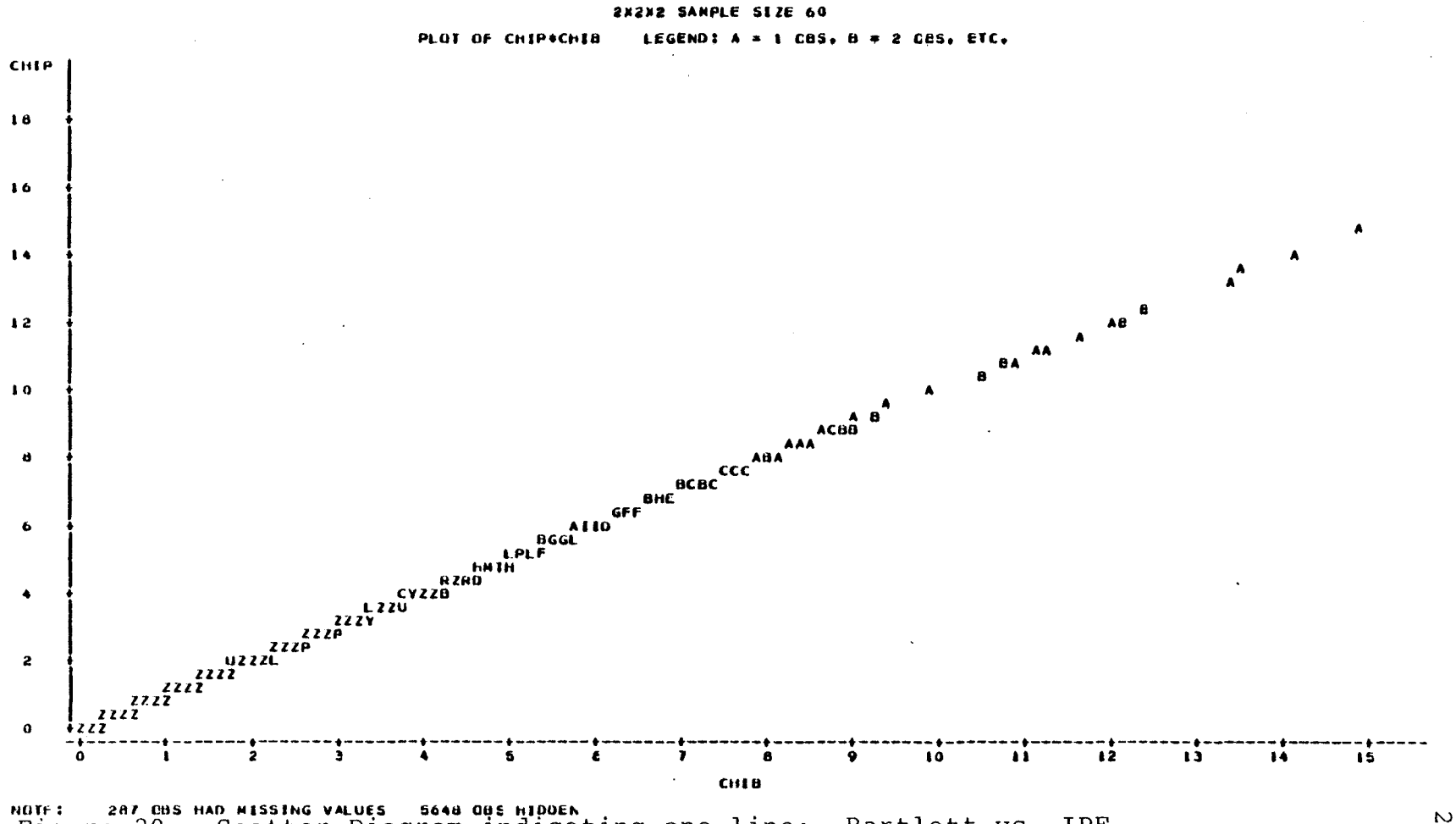

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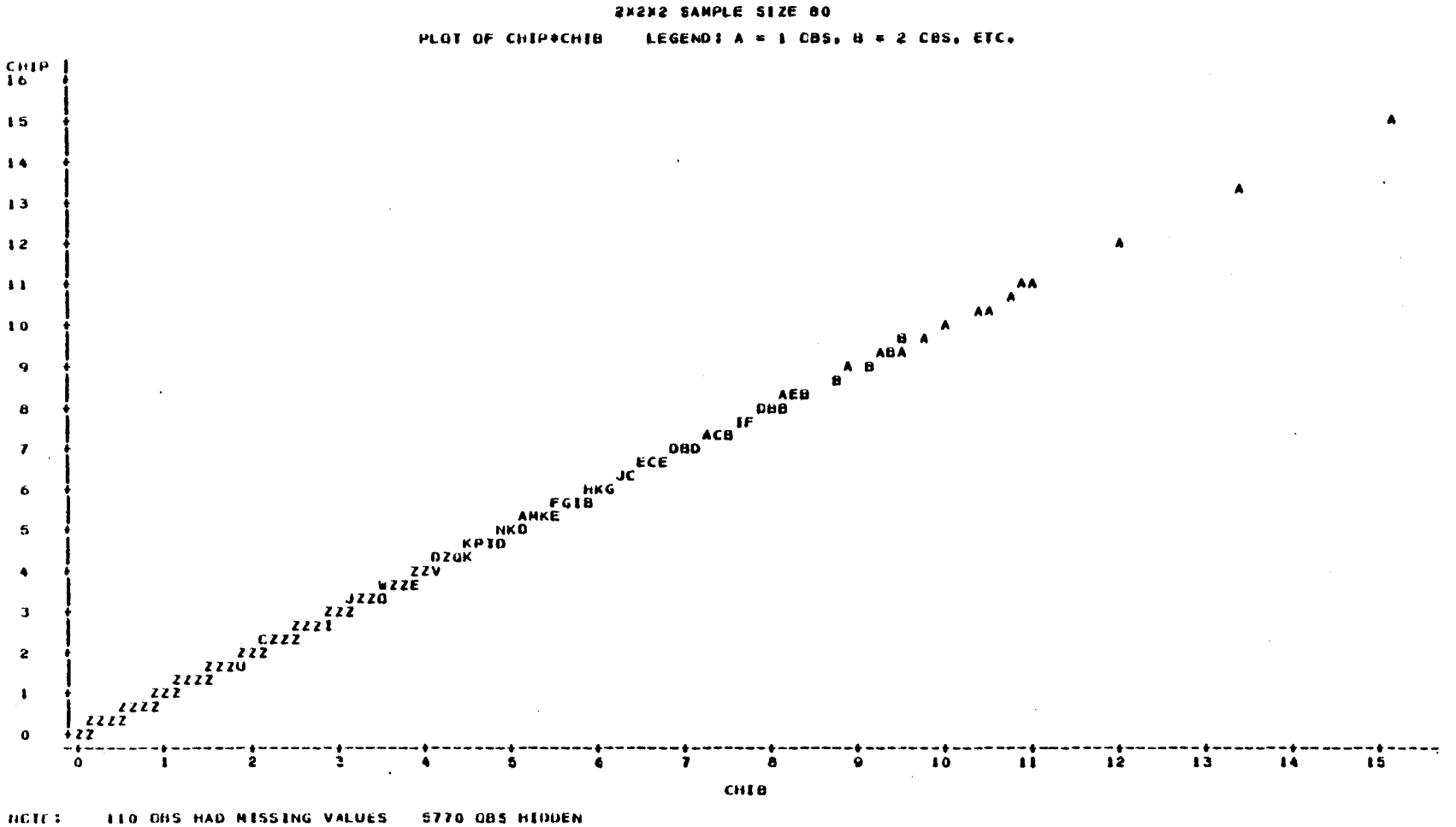

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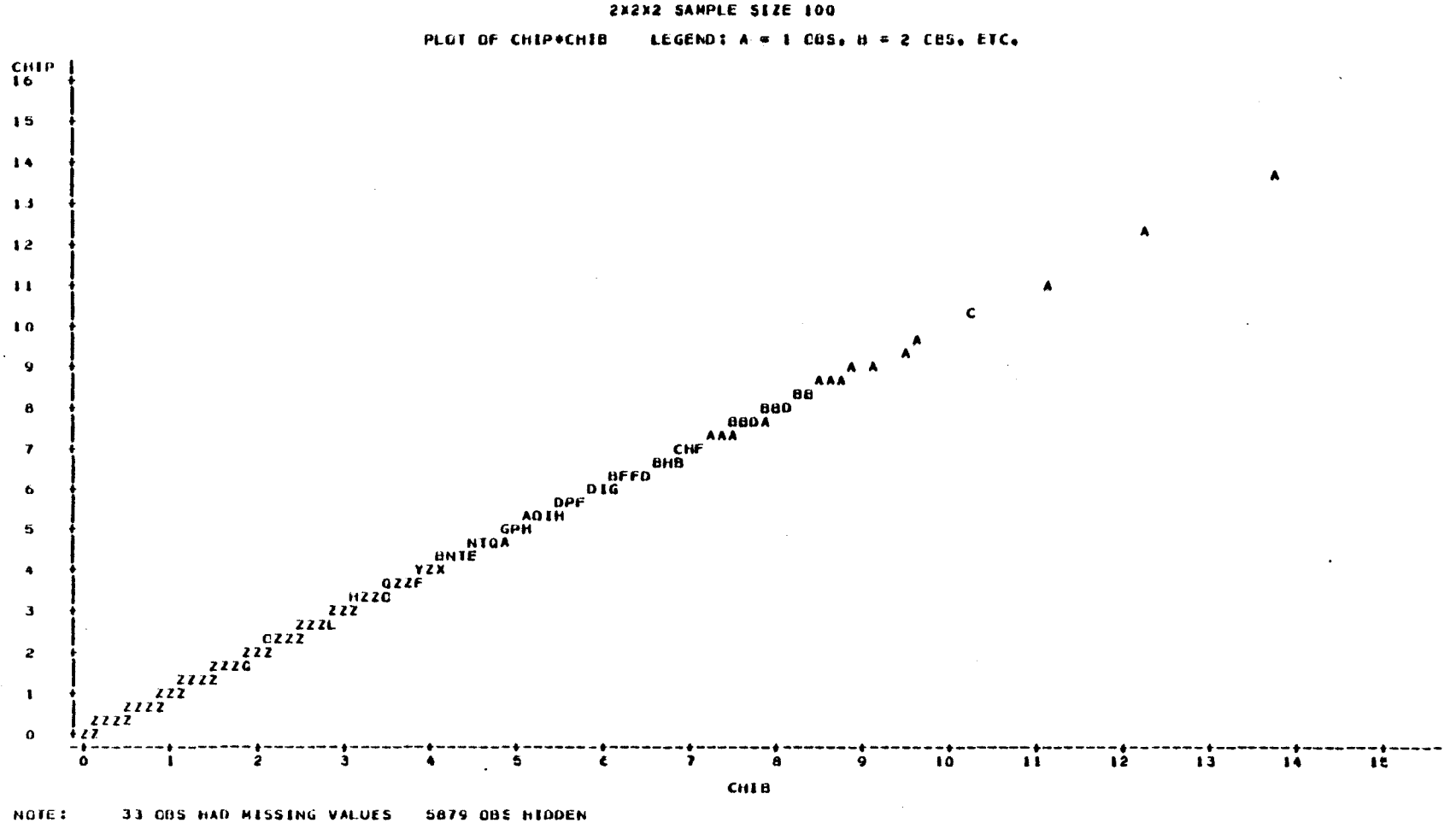


Figure 22. Scatter Diagram indicating one line: Bartlett vs. IPF

APPENDIX B

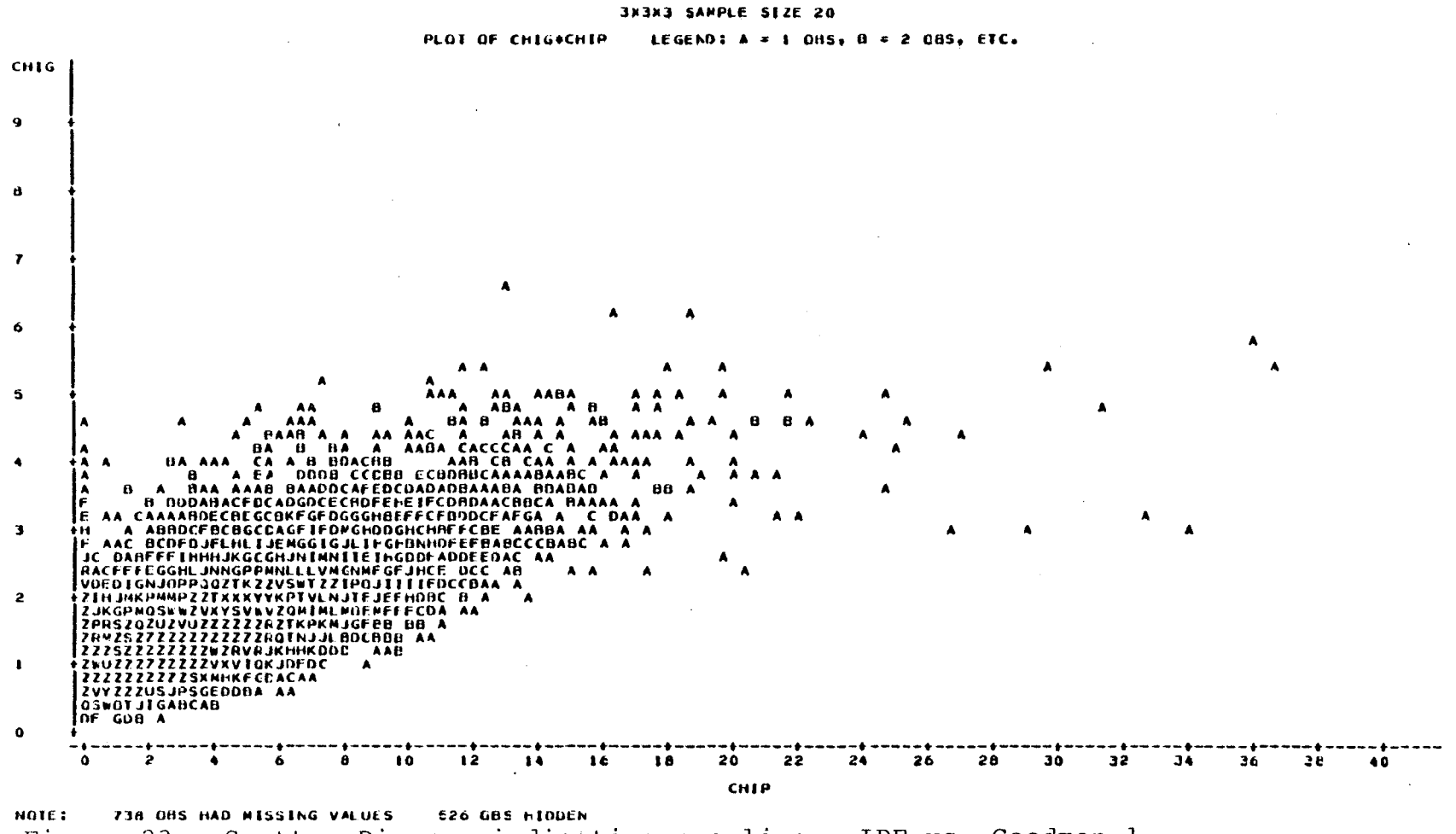
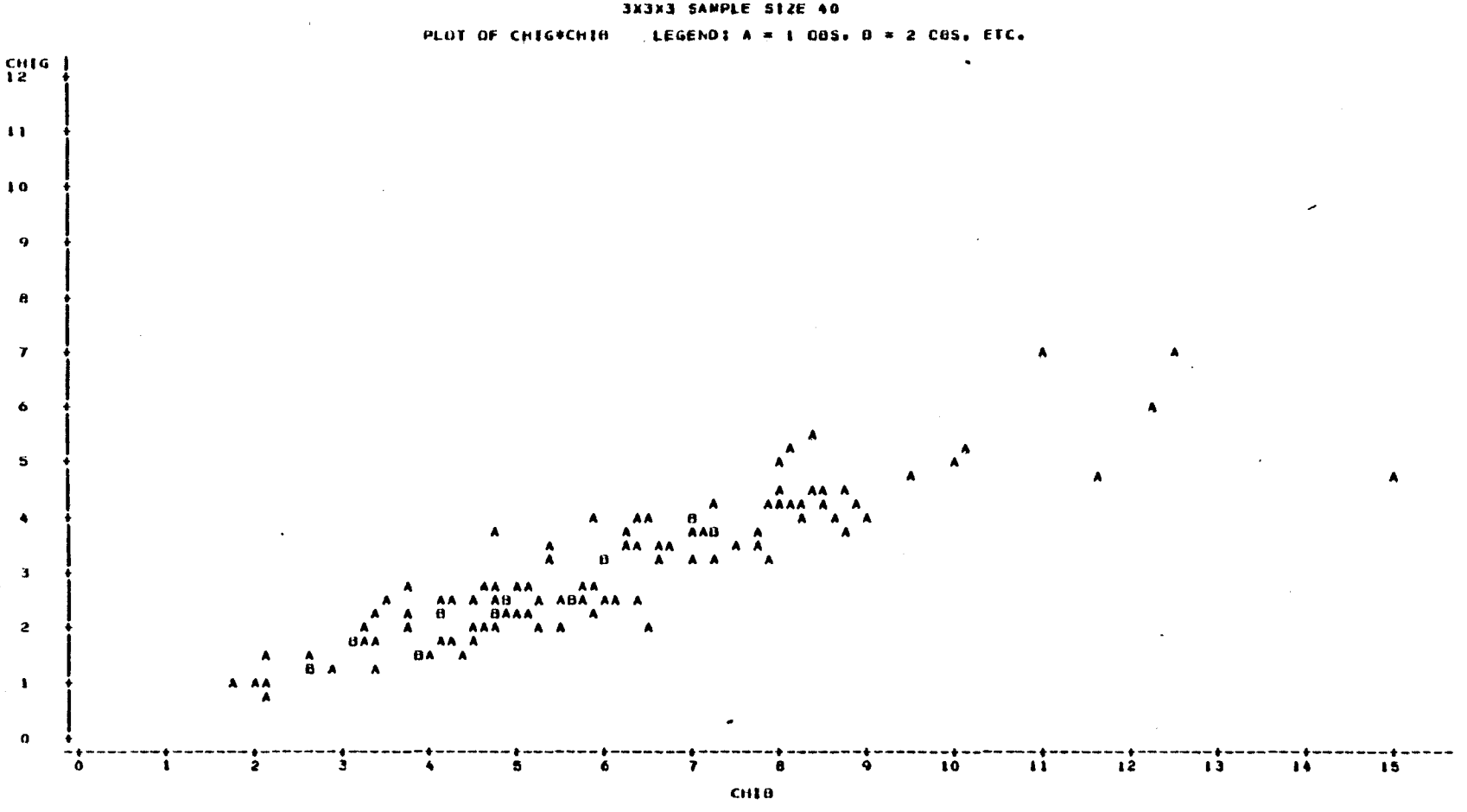


Figure 23. Scatter Diagram indicating one line: IPF vs. Goodman 1



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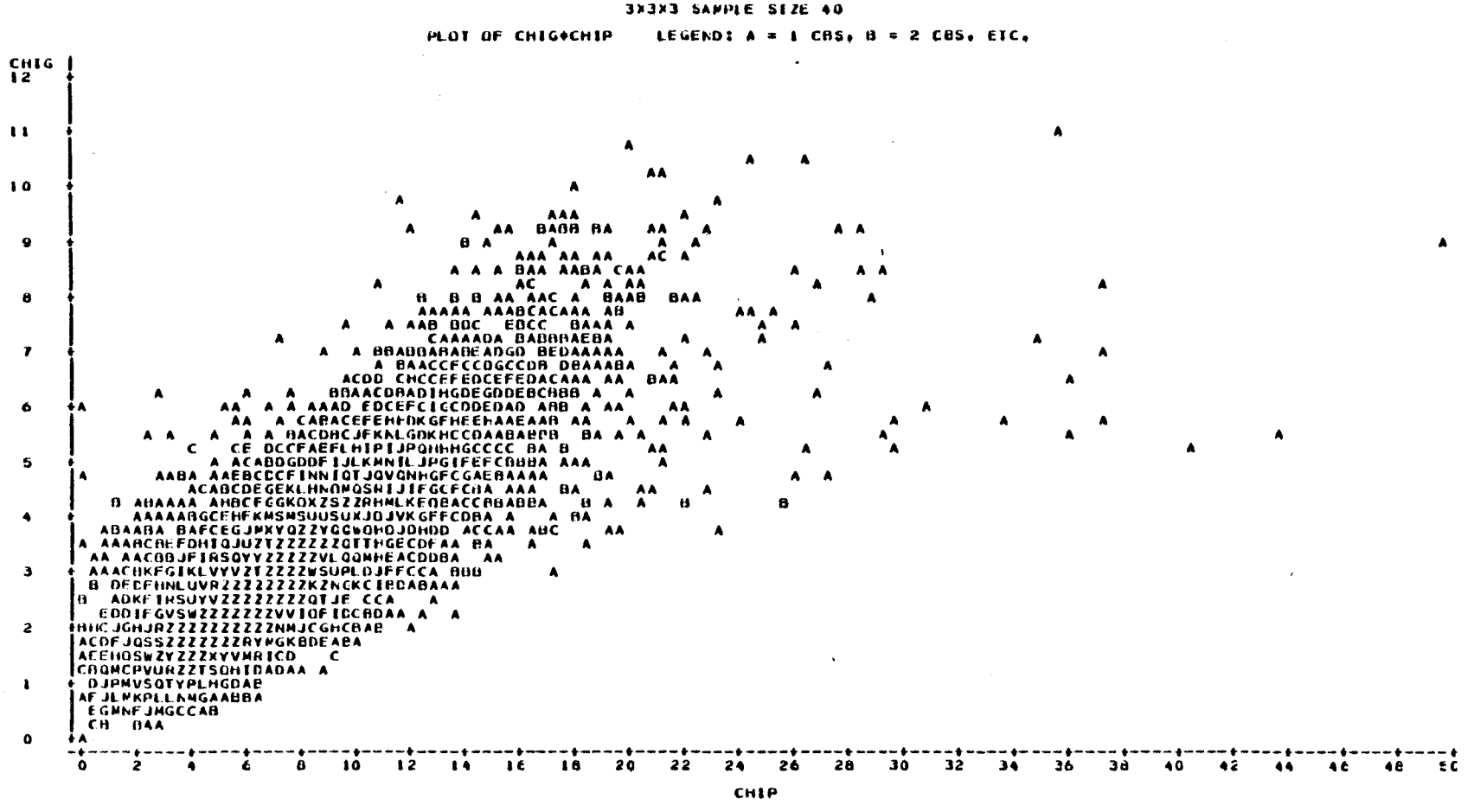

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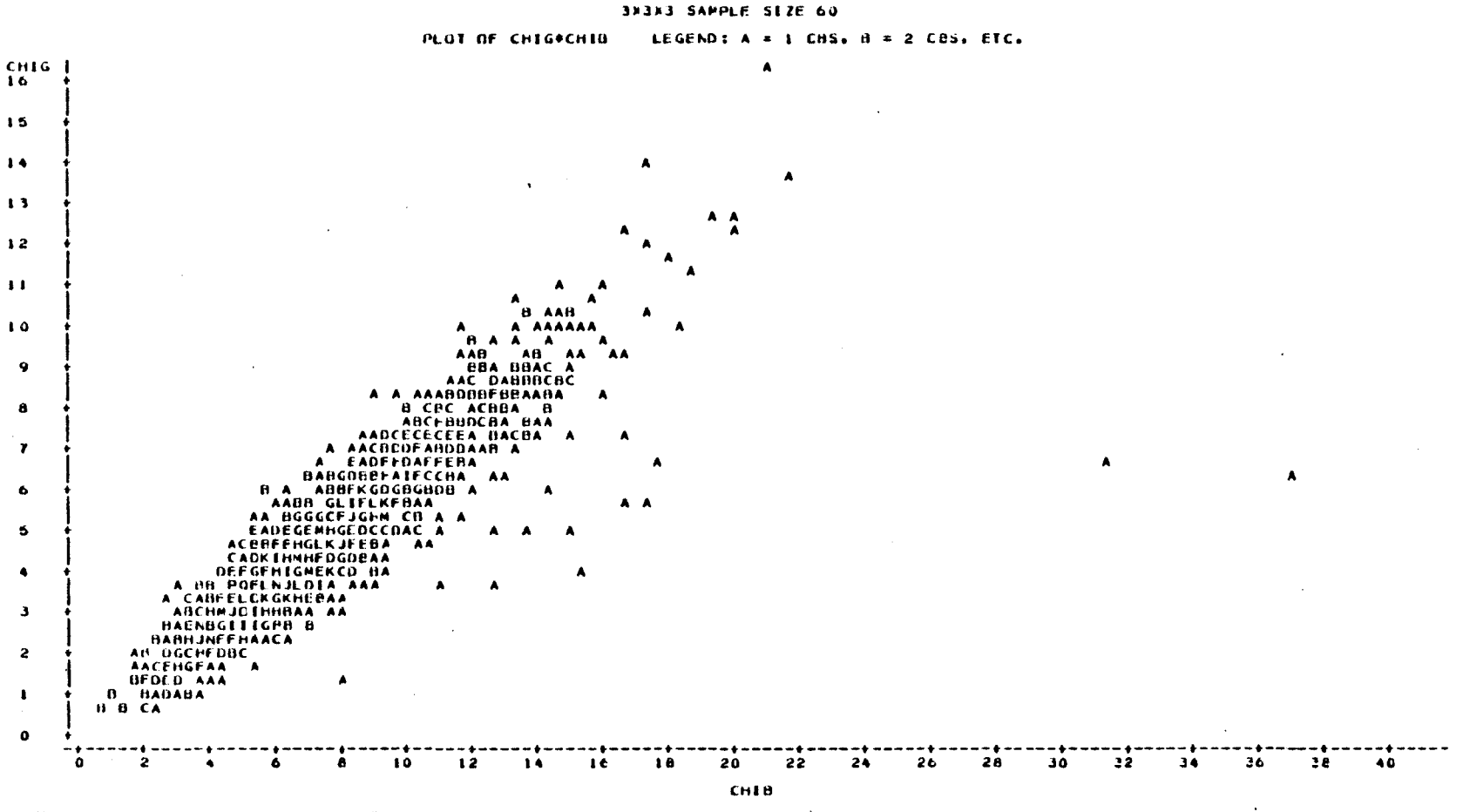

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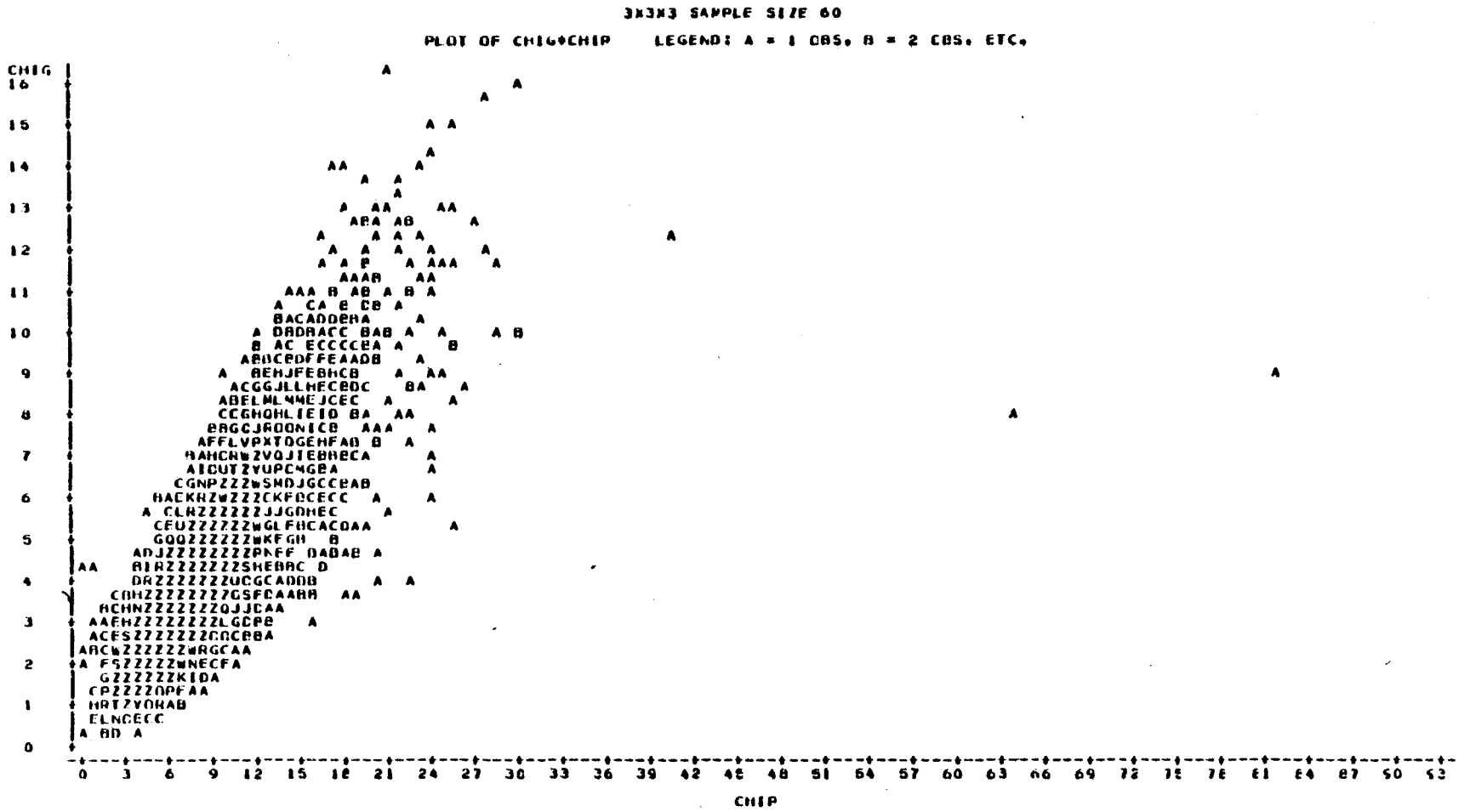

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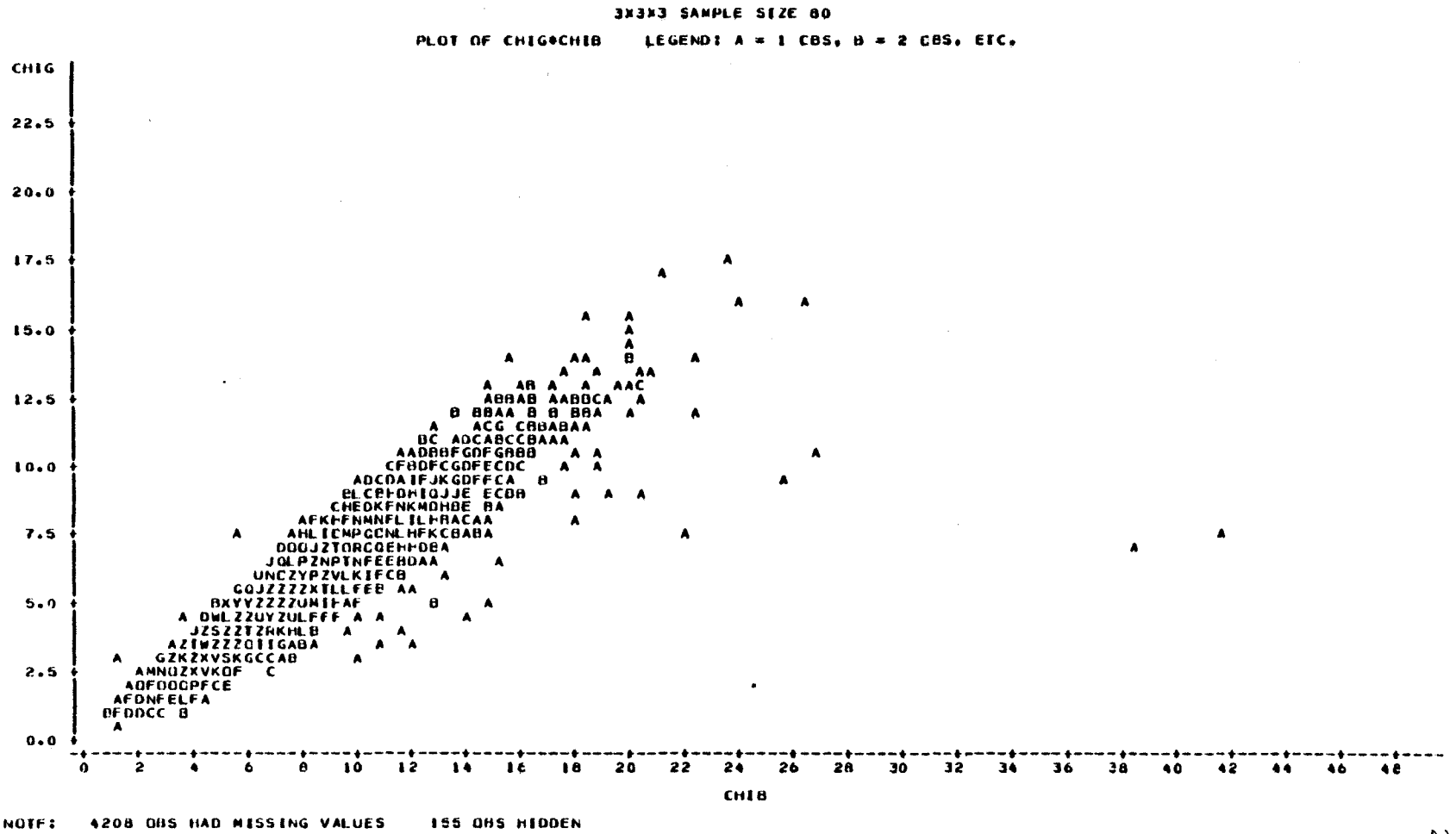

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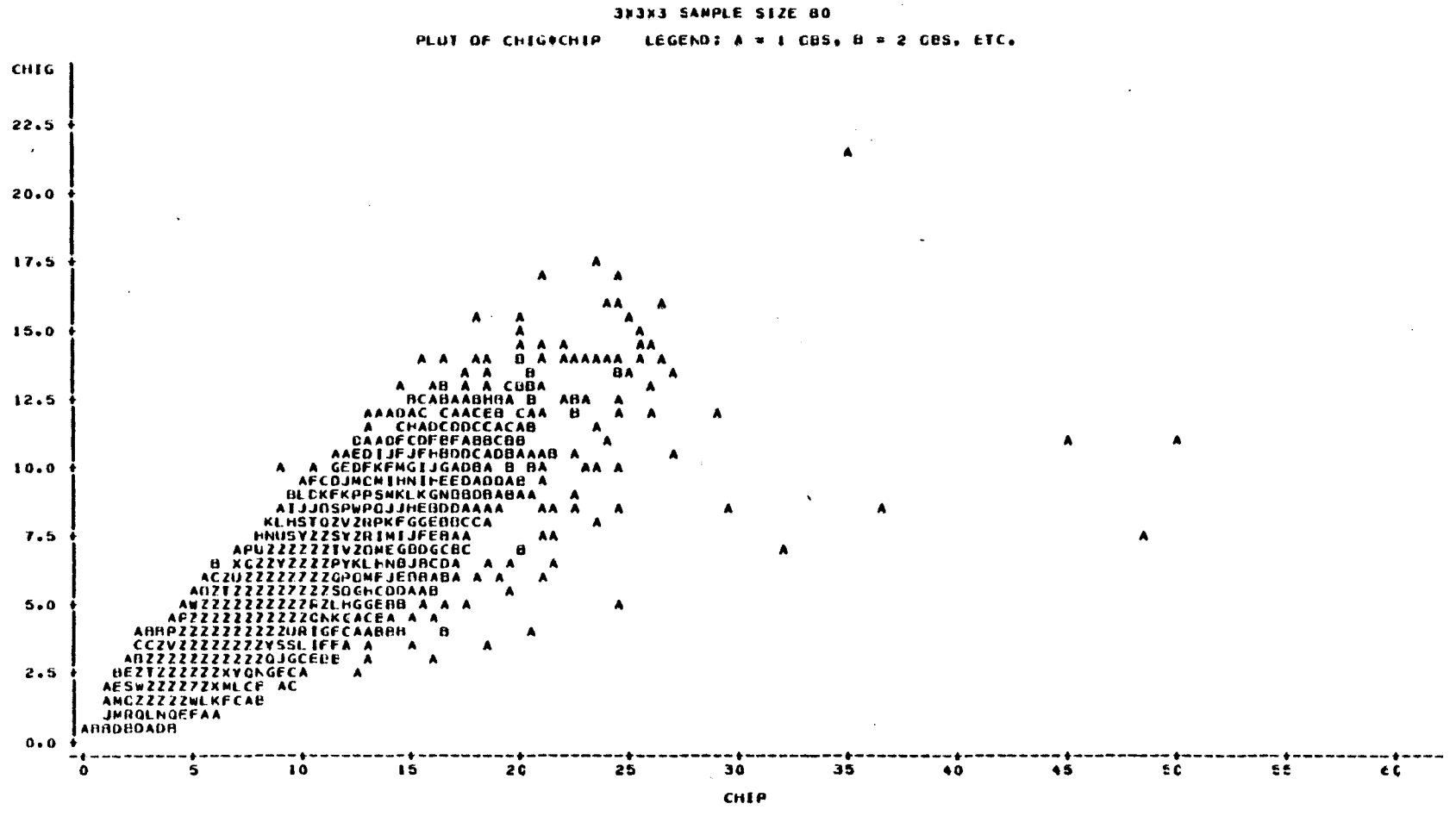

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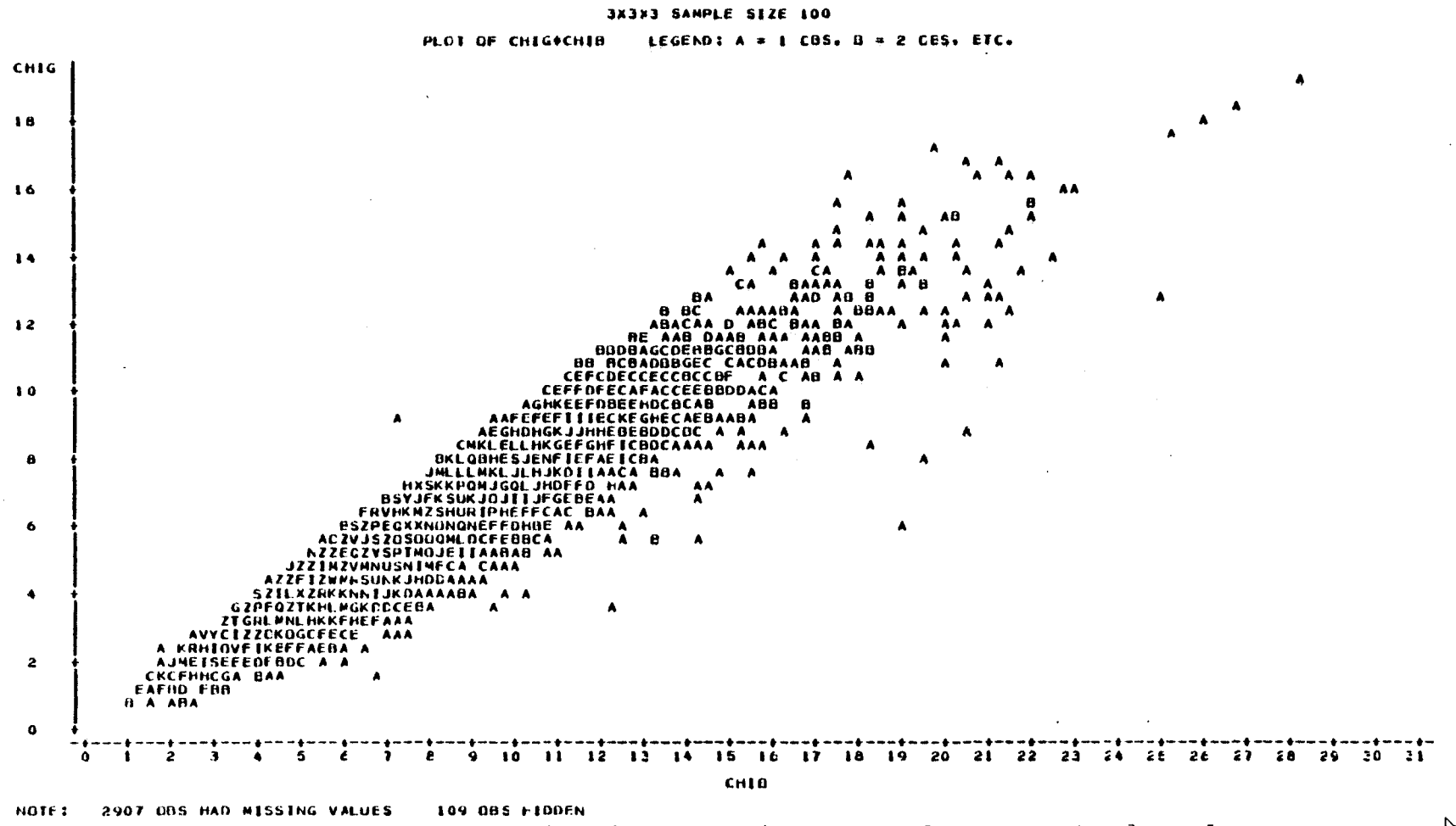

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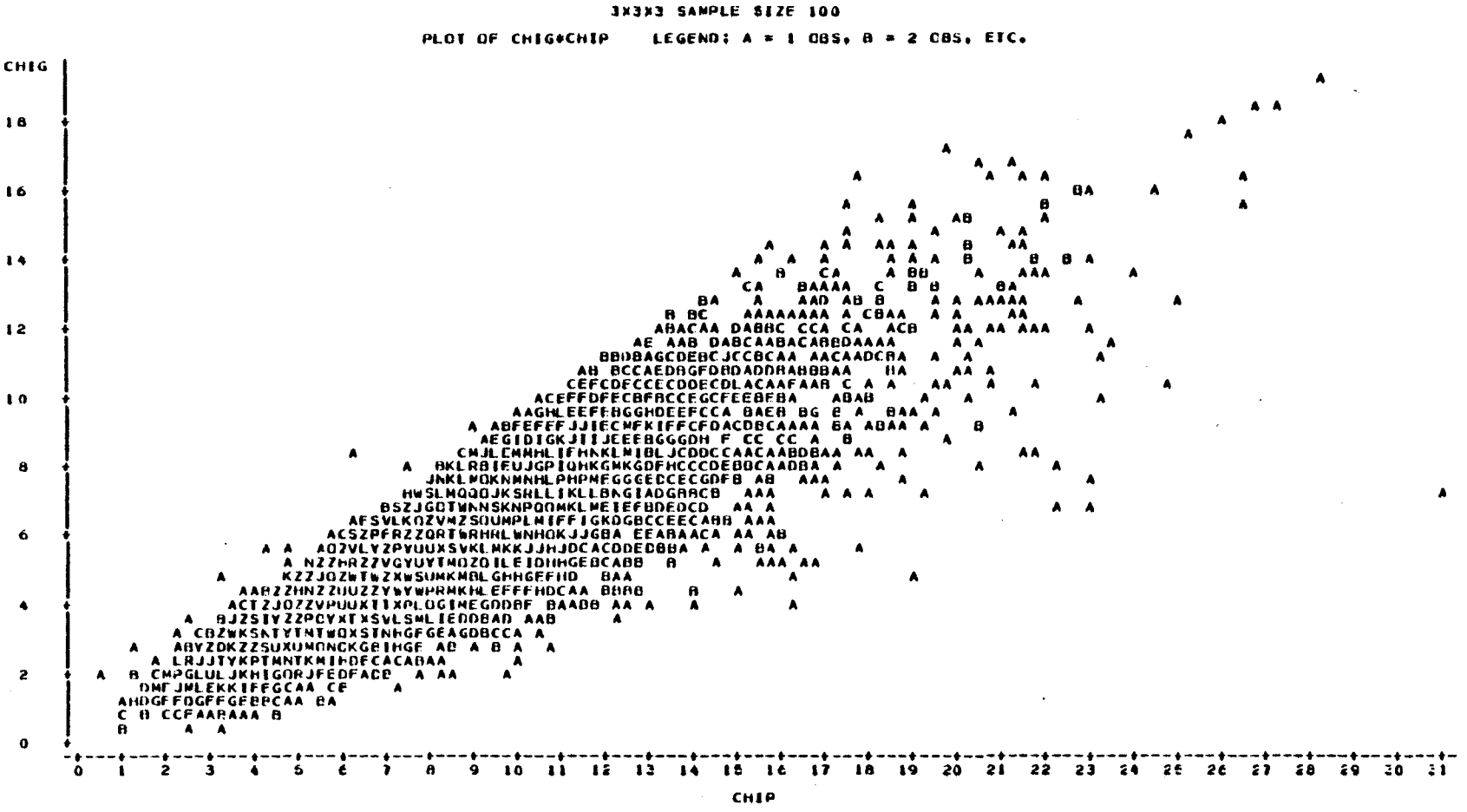

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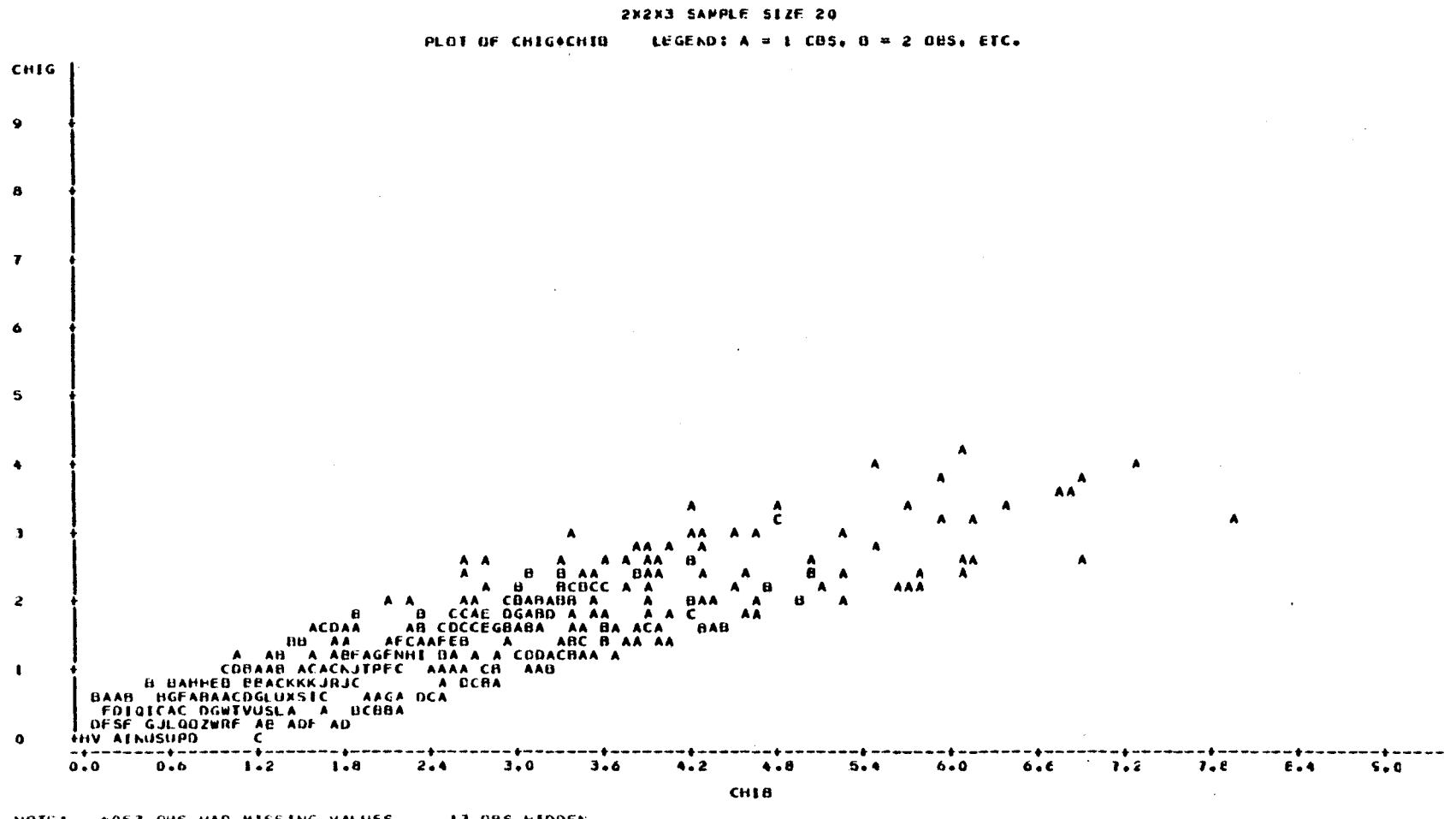

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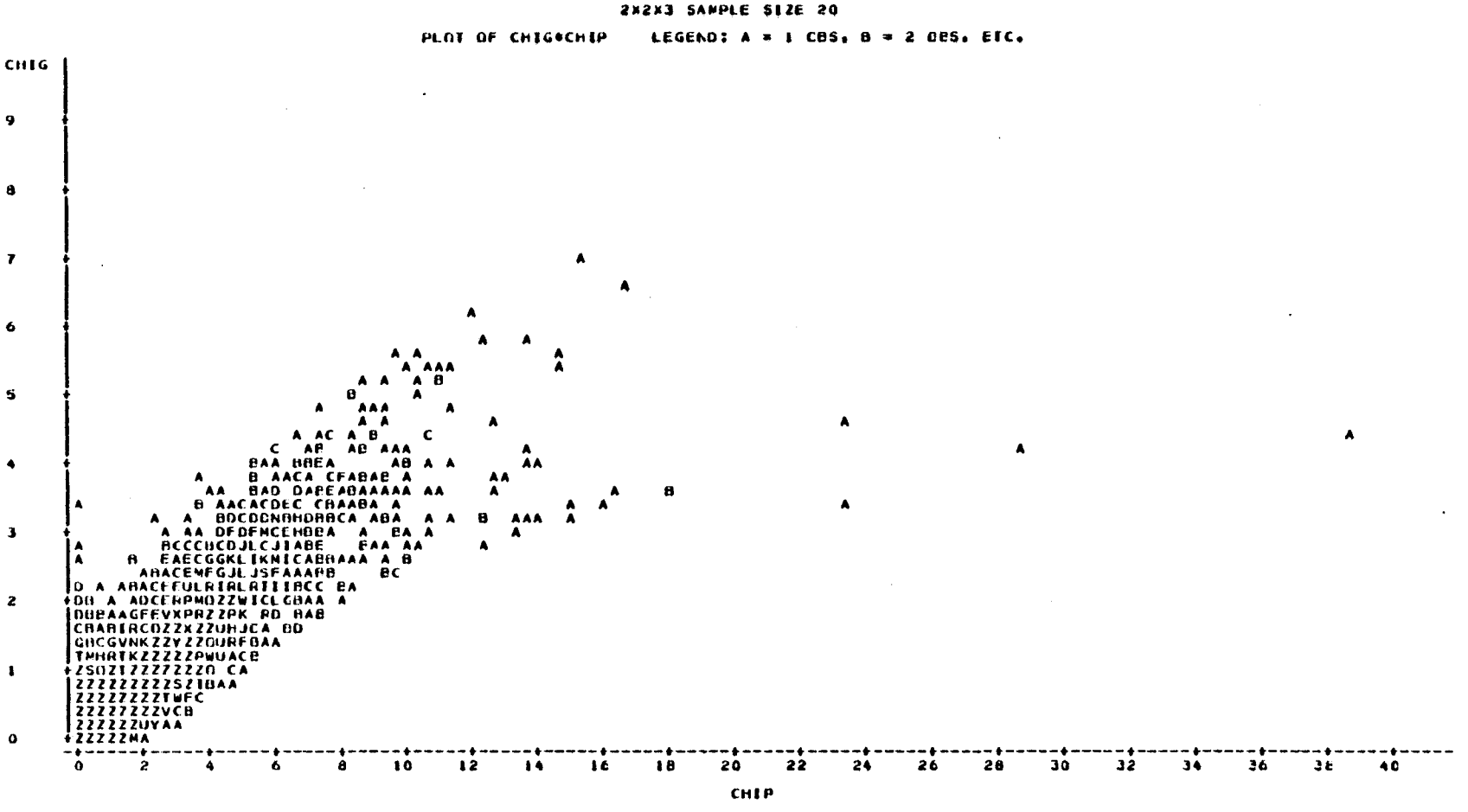

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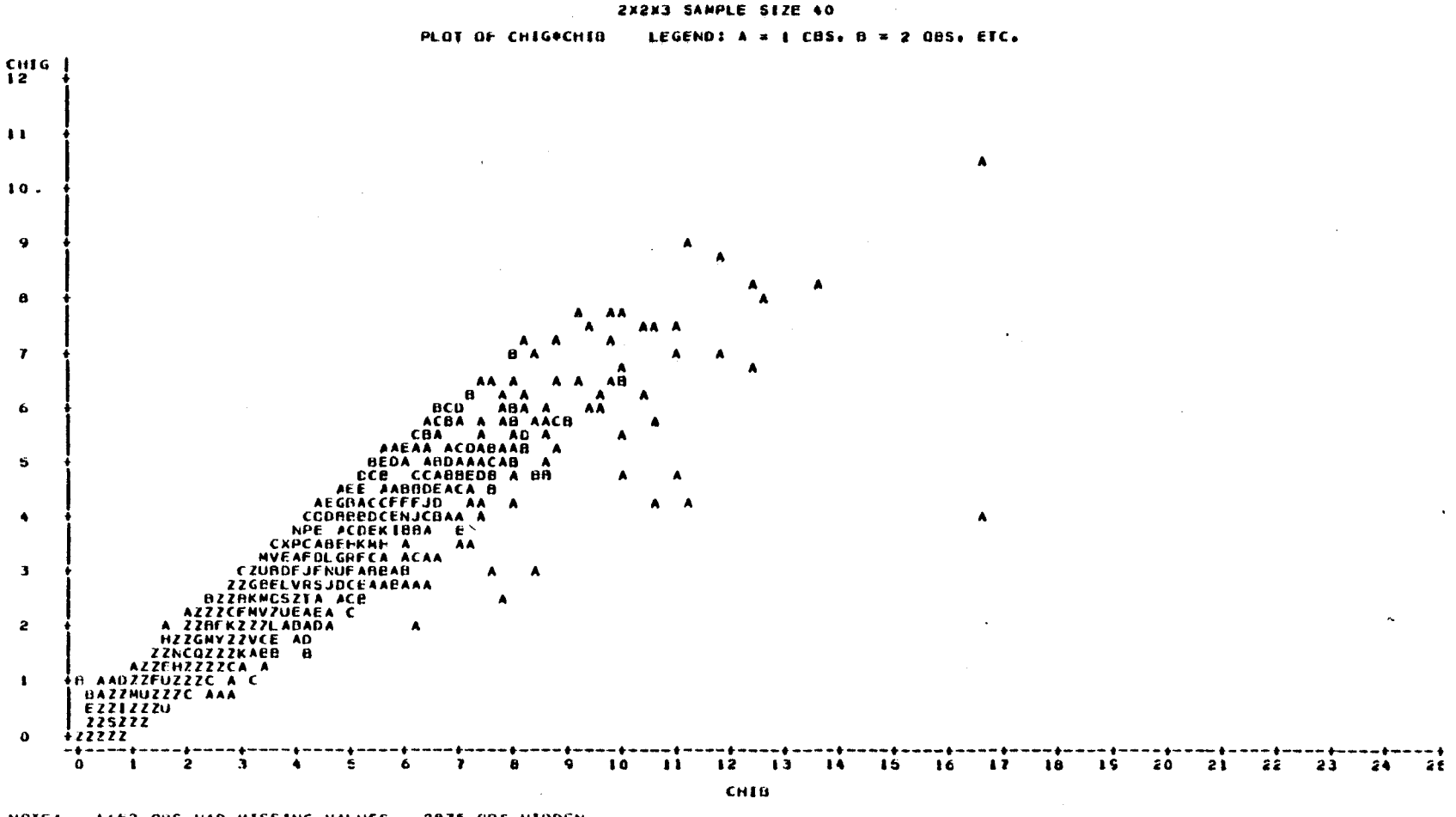

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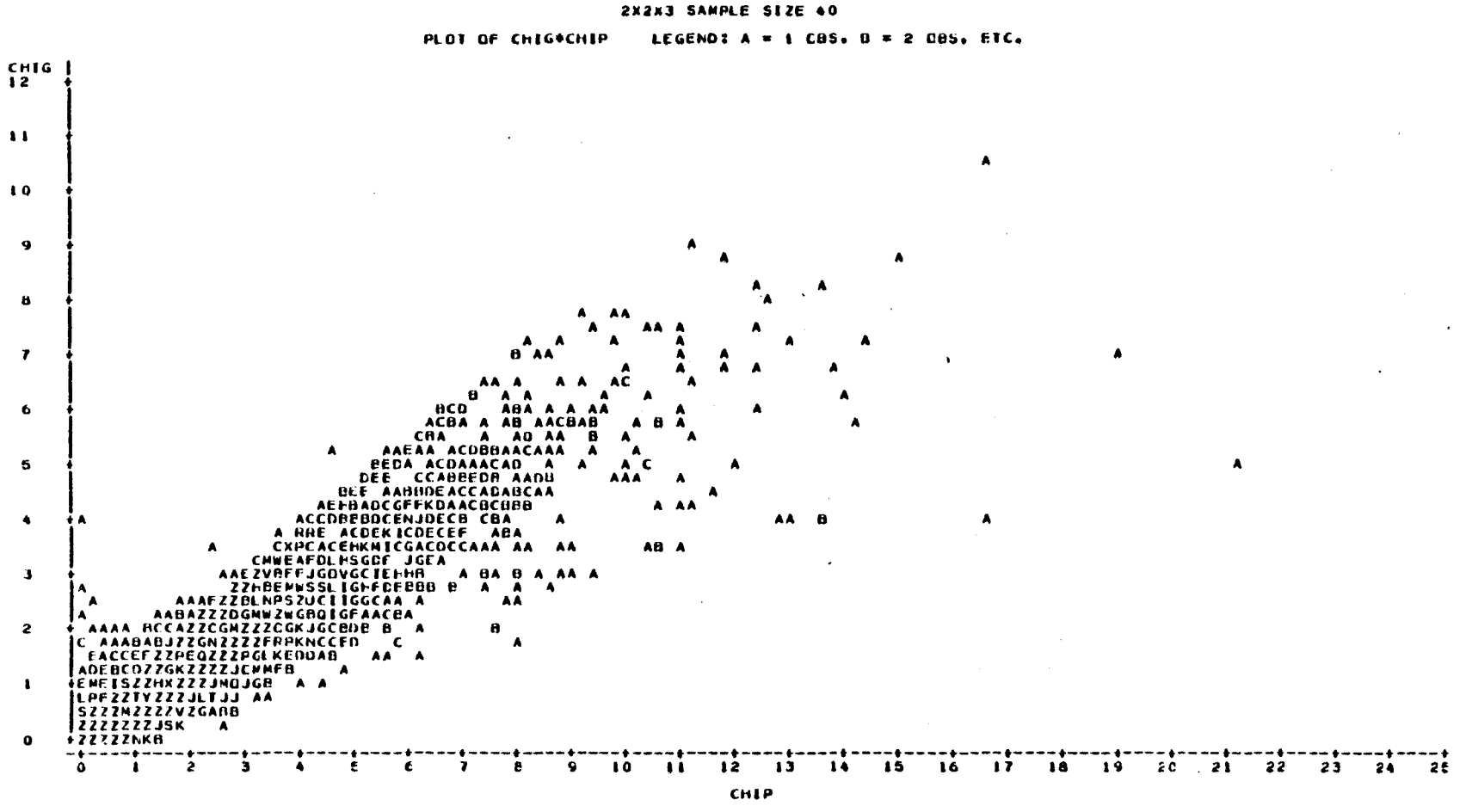

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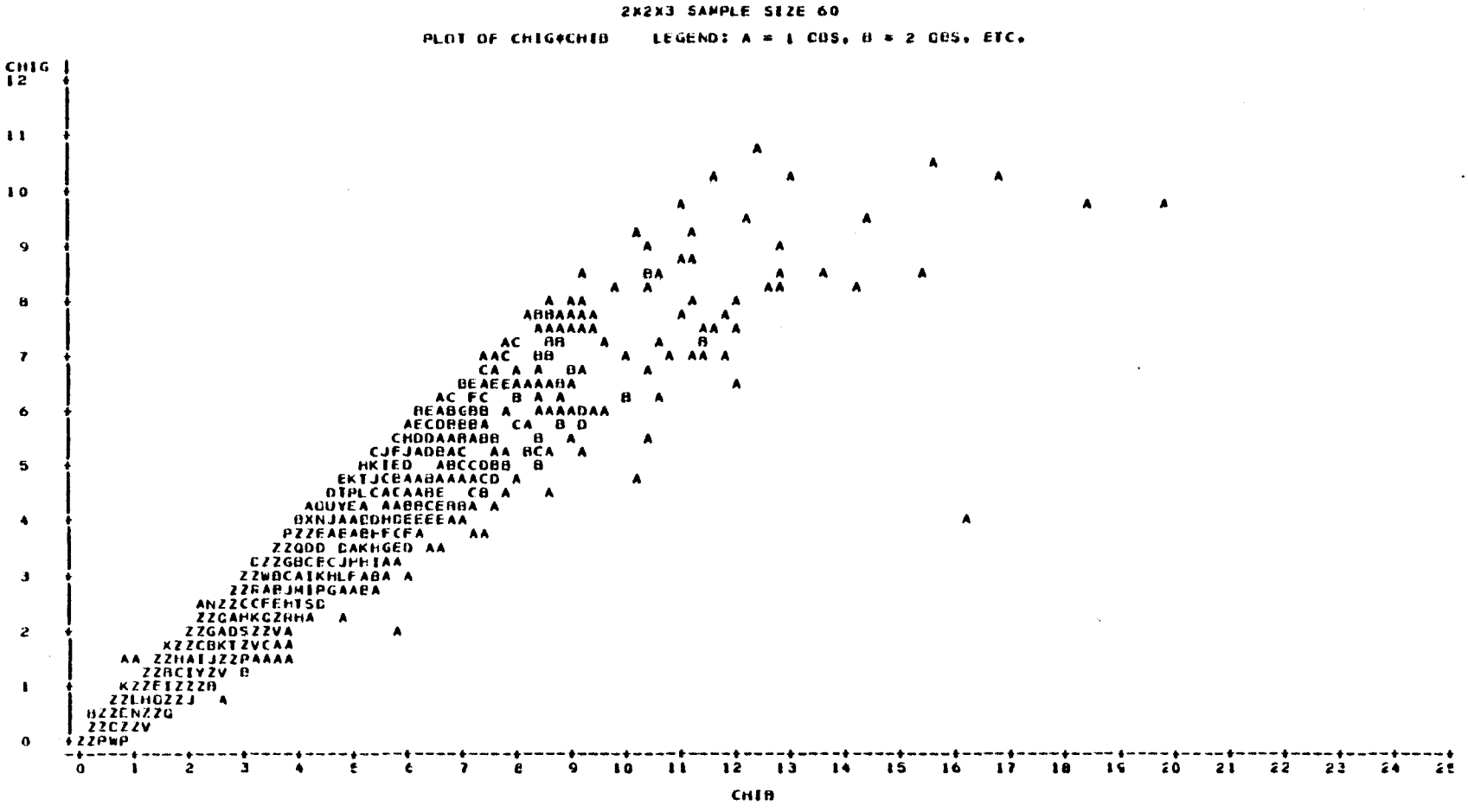

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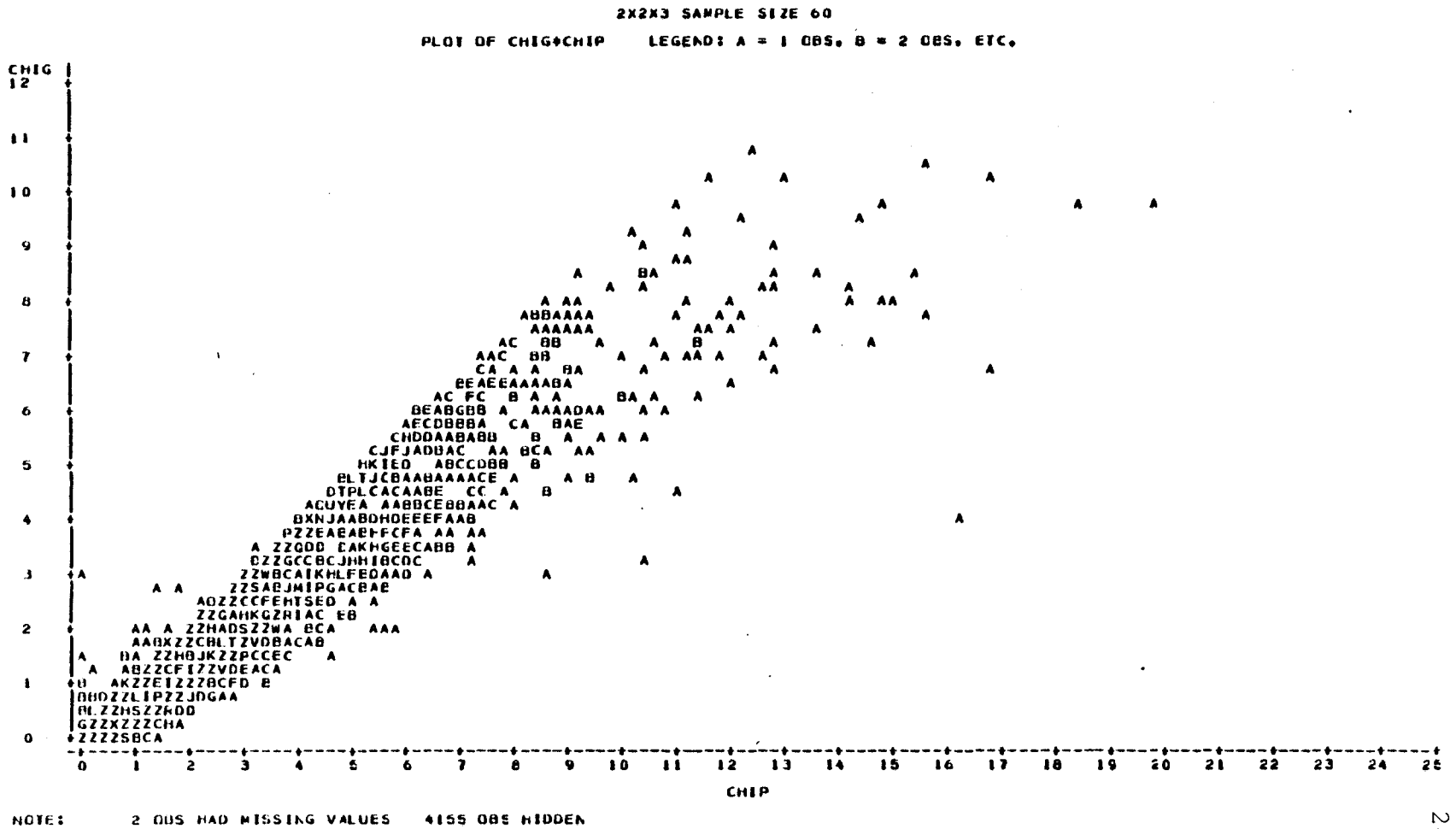

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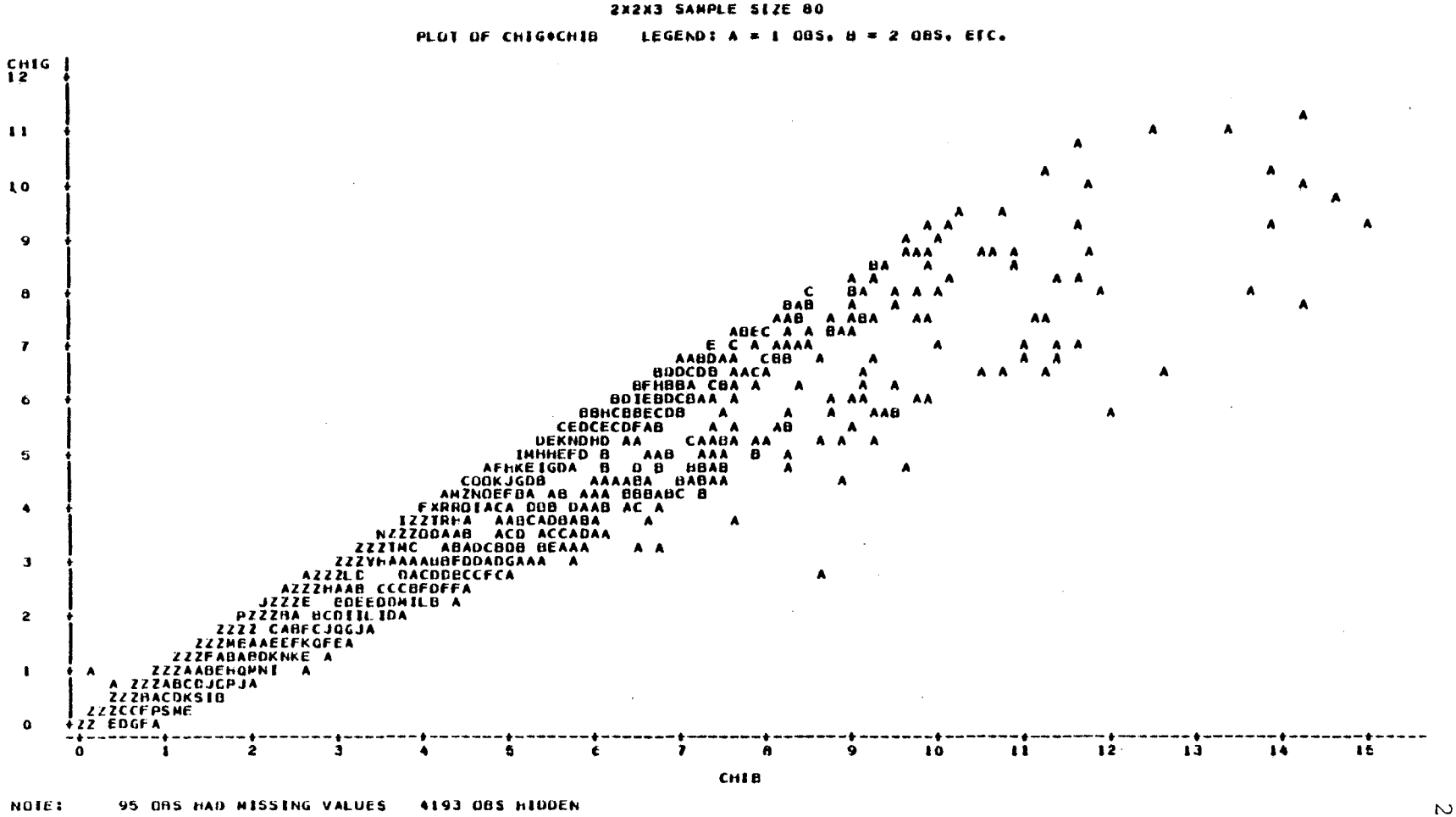

Figure 38. Scatter Diagram indicating one line: Bartlett vs. Goodman 1

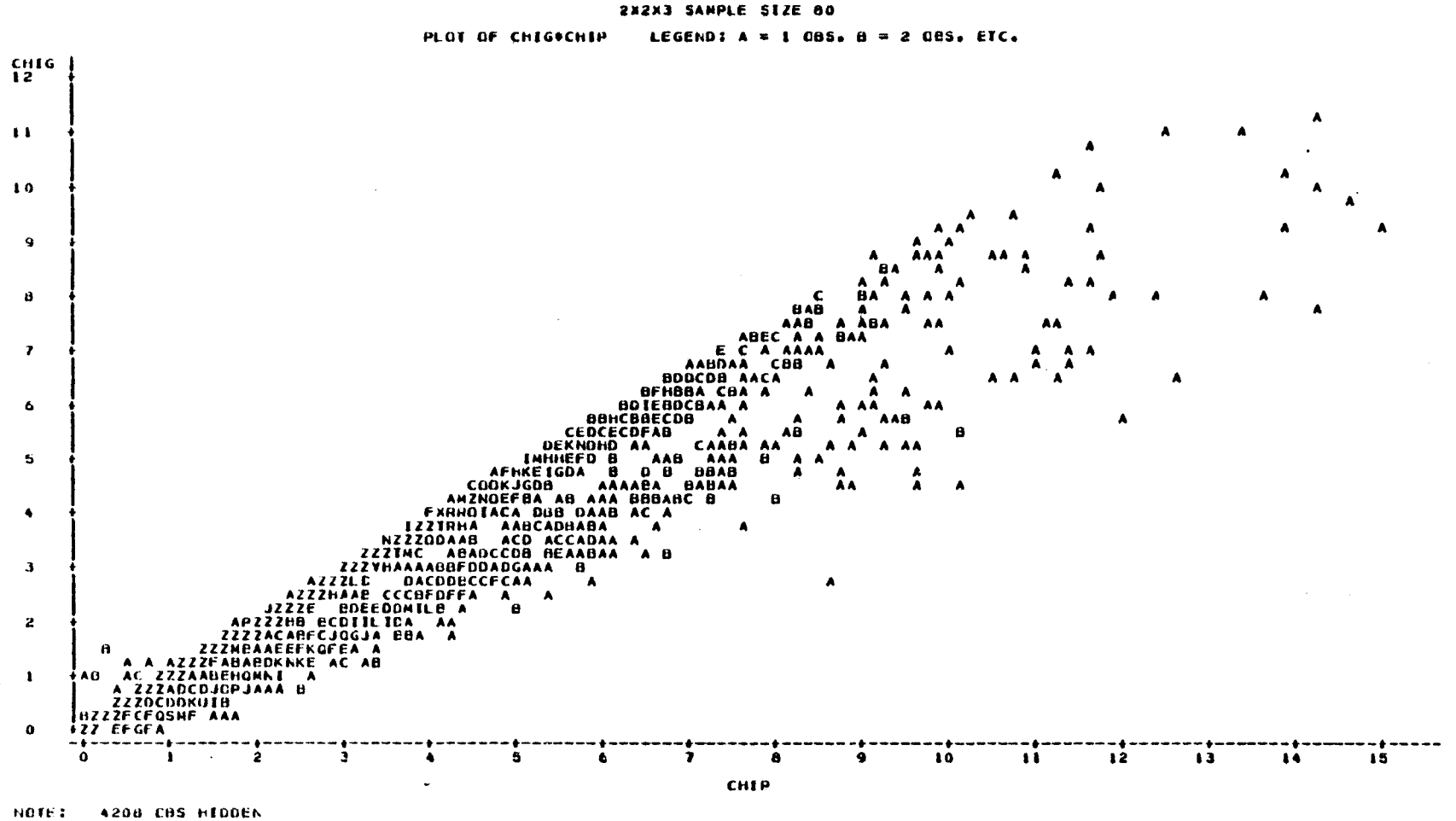

Figure 39. Scatter Diagram indicating one line: IPF vs. Goodman 1

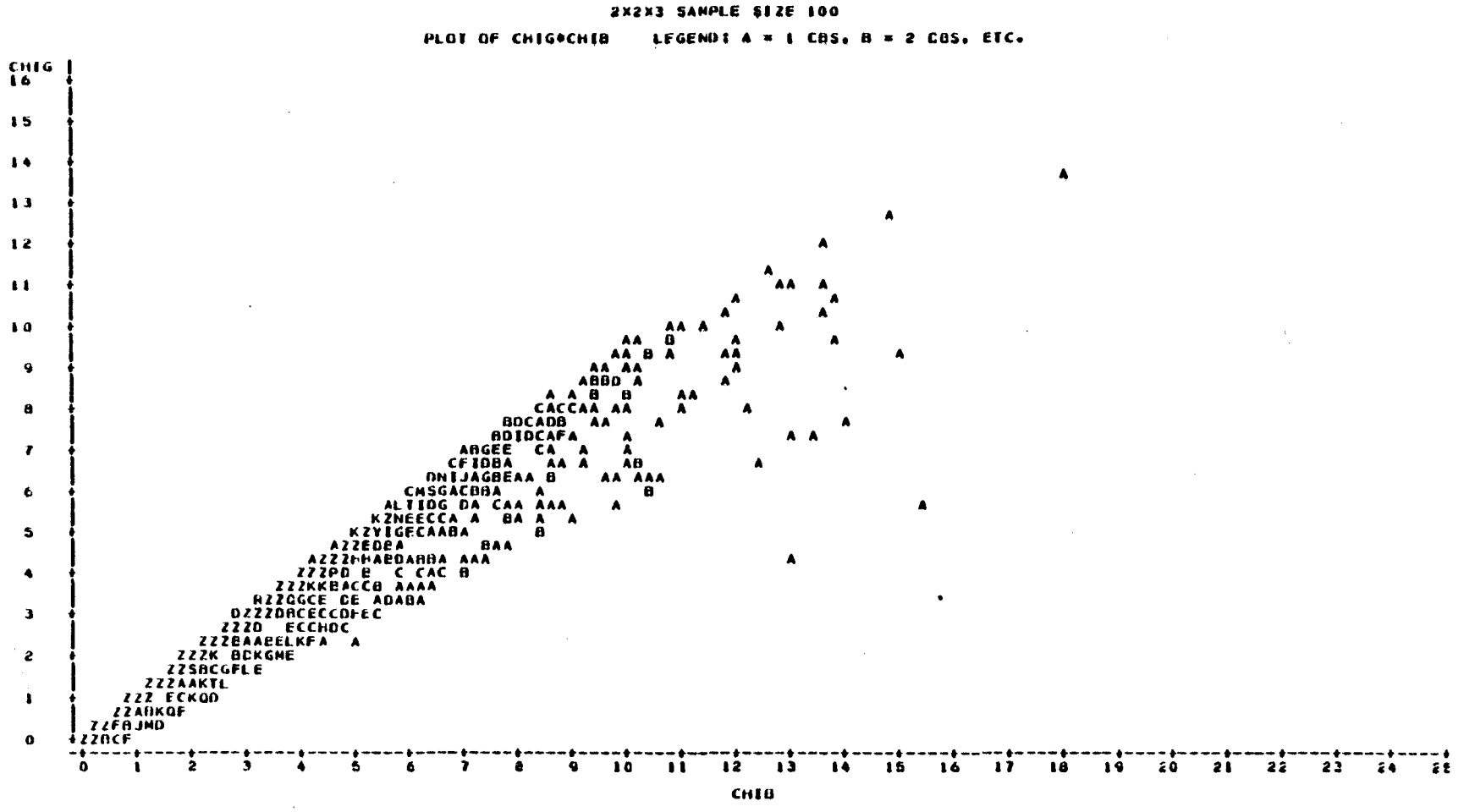

Figure 40. Scatter Diagram indicating one line: Bartlett vs. Goodman 1

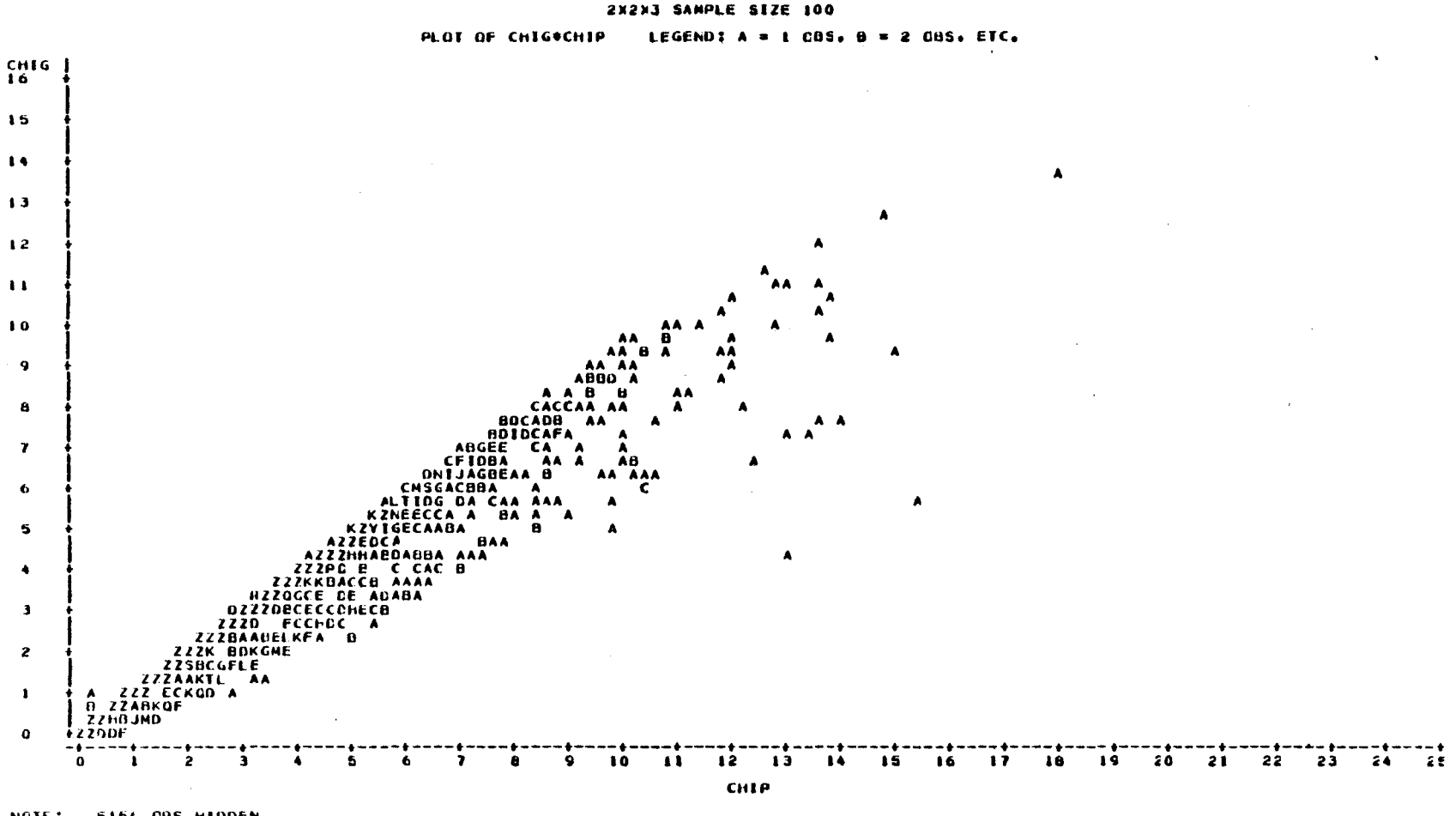

Figure 41. Scatter Diagram indicating one line: IPF vs. Goodman 1

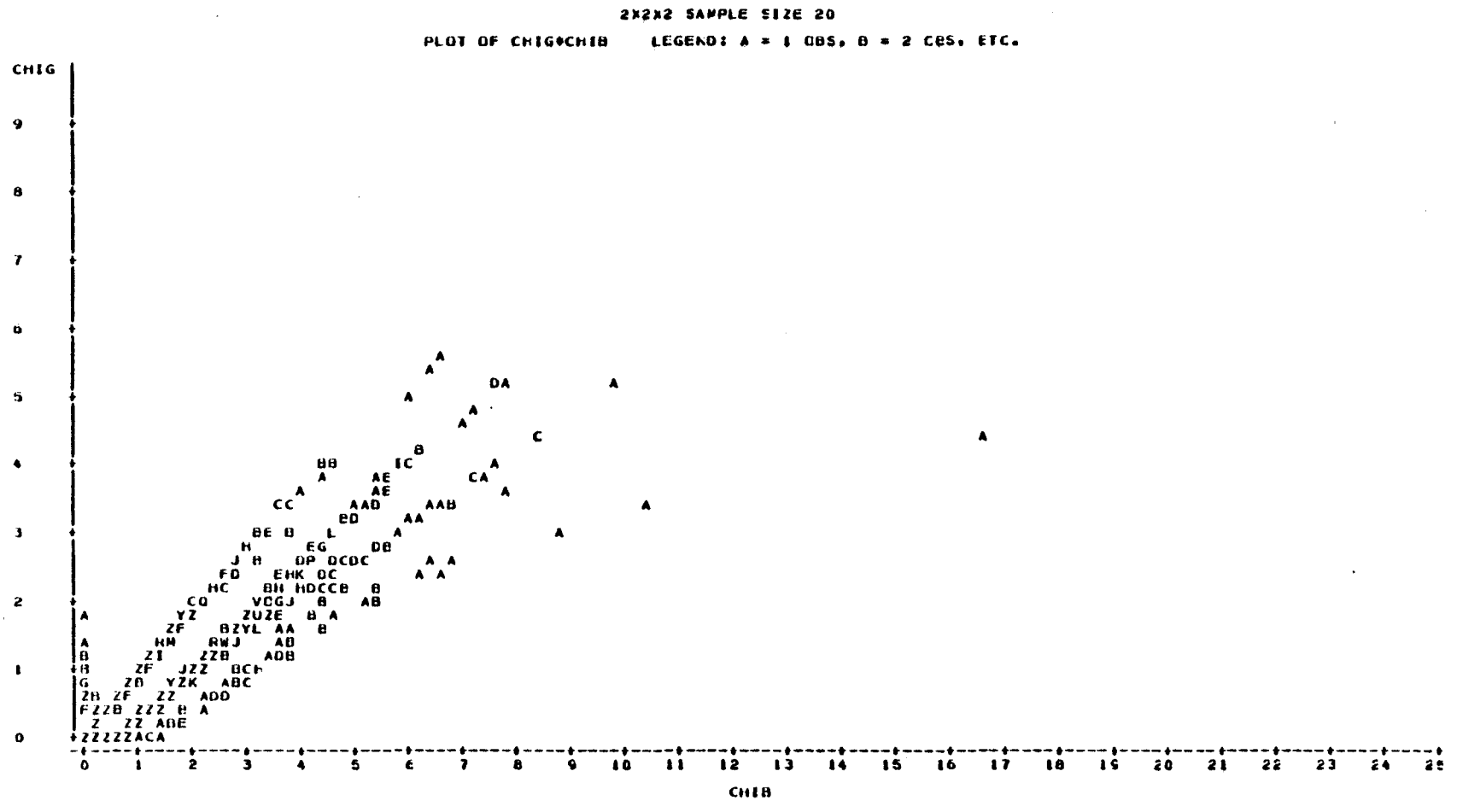

Figure 42. Scatter Diagram indicating two or three lines: Bartlett vs. Goodman 1

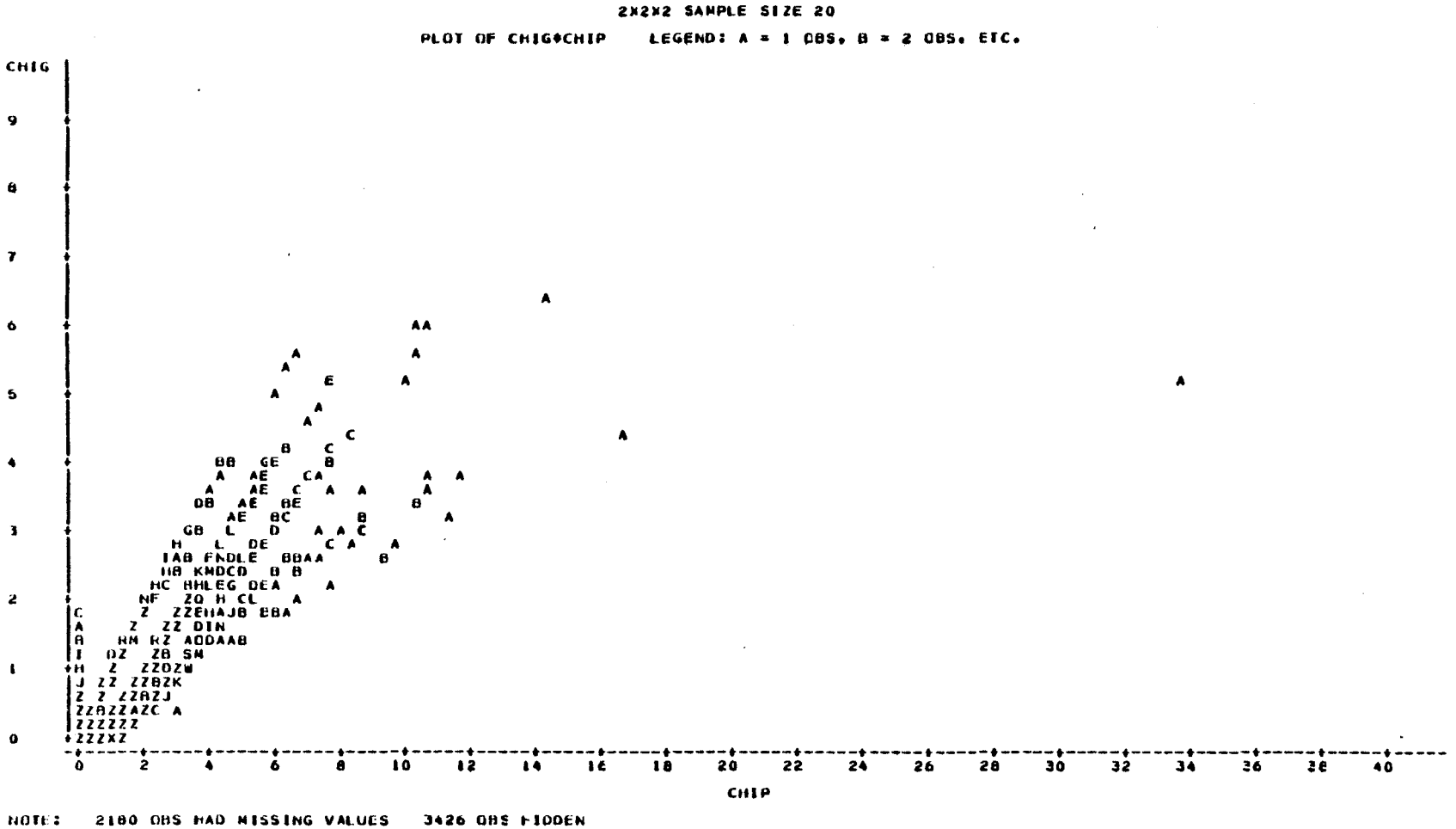

Figure 43. Scatter Diagram indicating two or three lines: IPF vs. Goodman 1

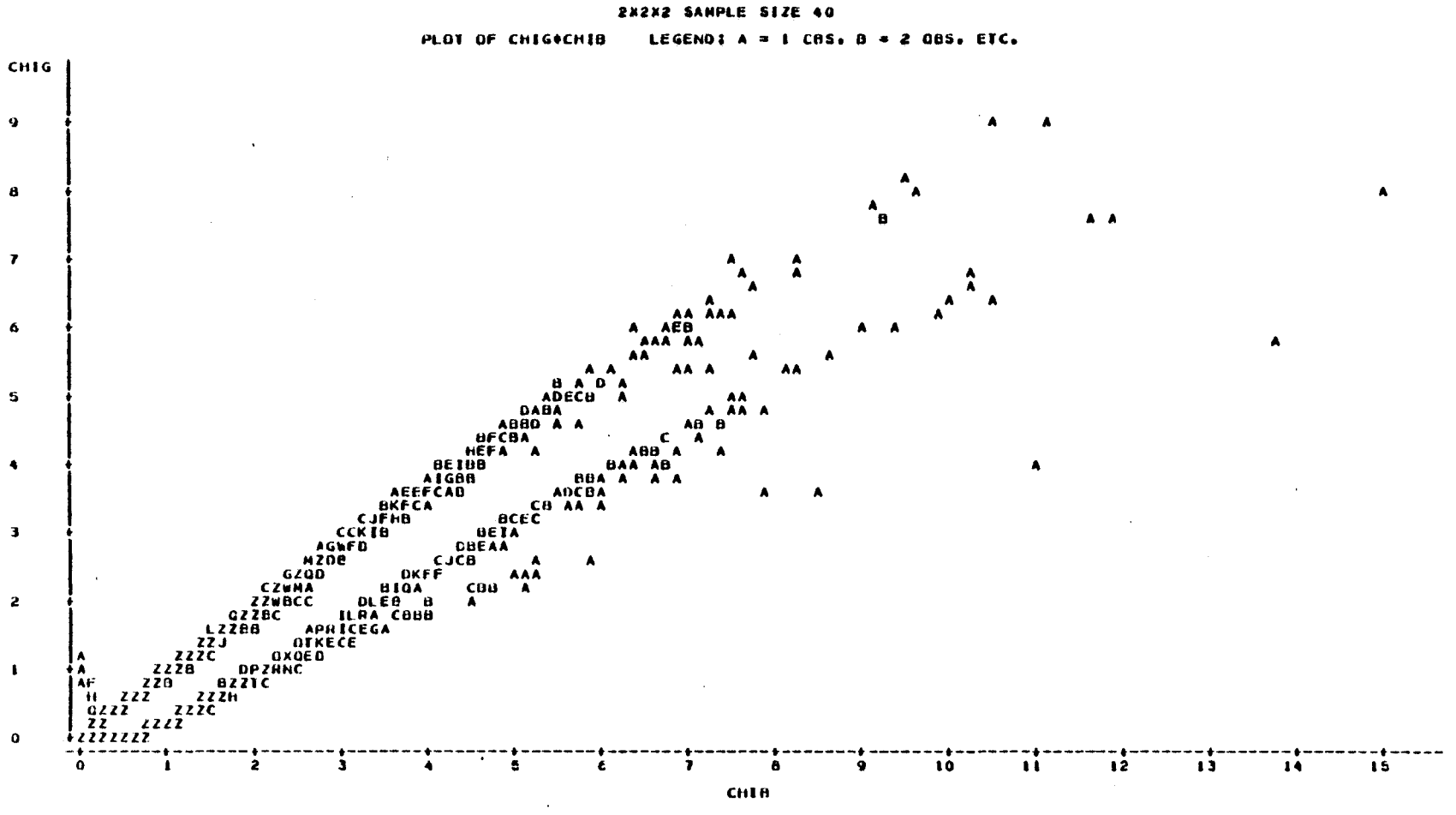

Figure 44. Scatter Diagram indicating two or three lines: Bartlett vs. Goodman 1

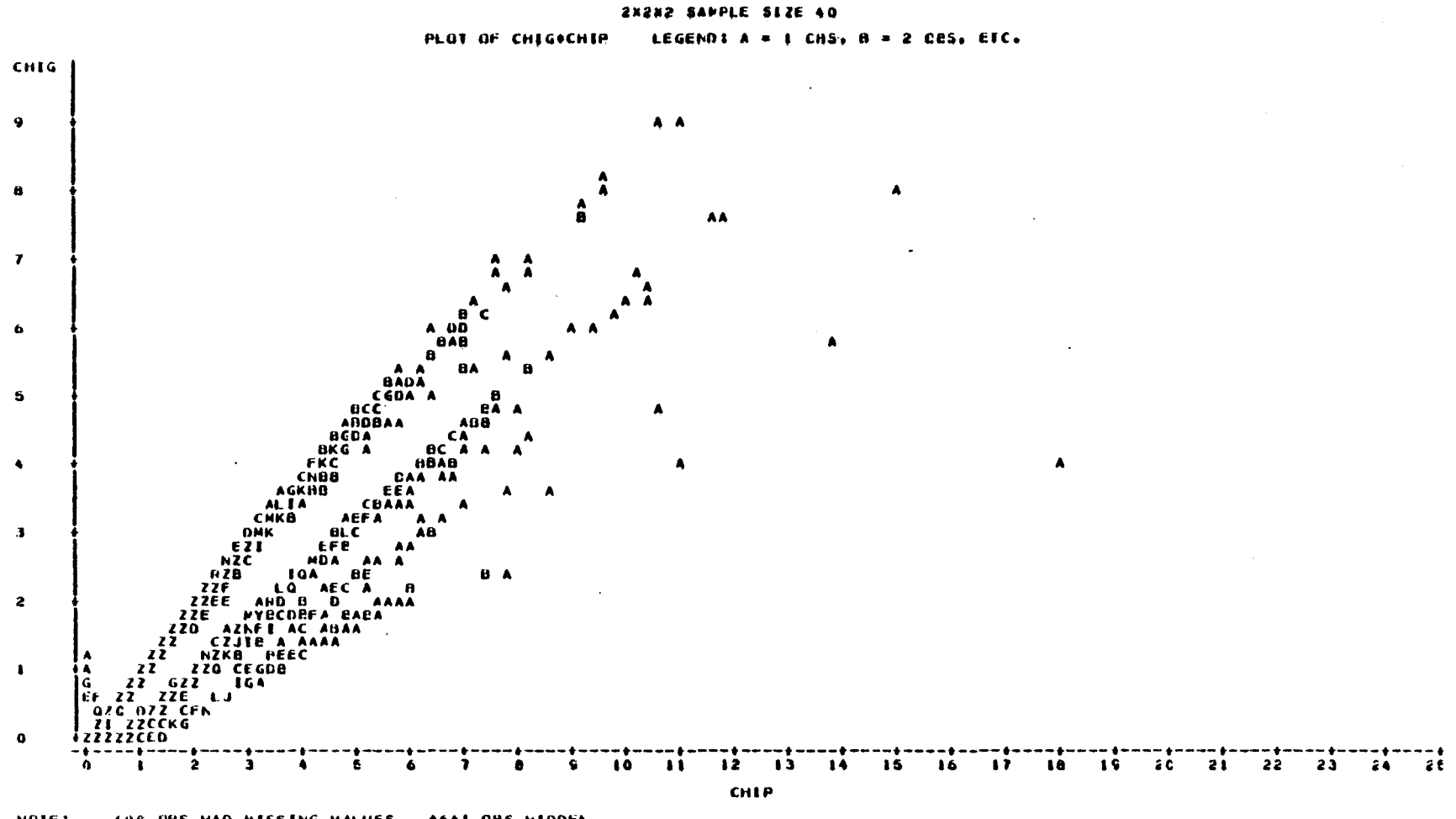

Figure 45. Scatter Diagram indicating two or three lines: IPF vs. Goodman's

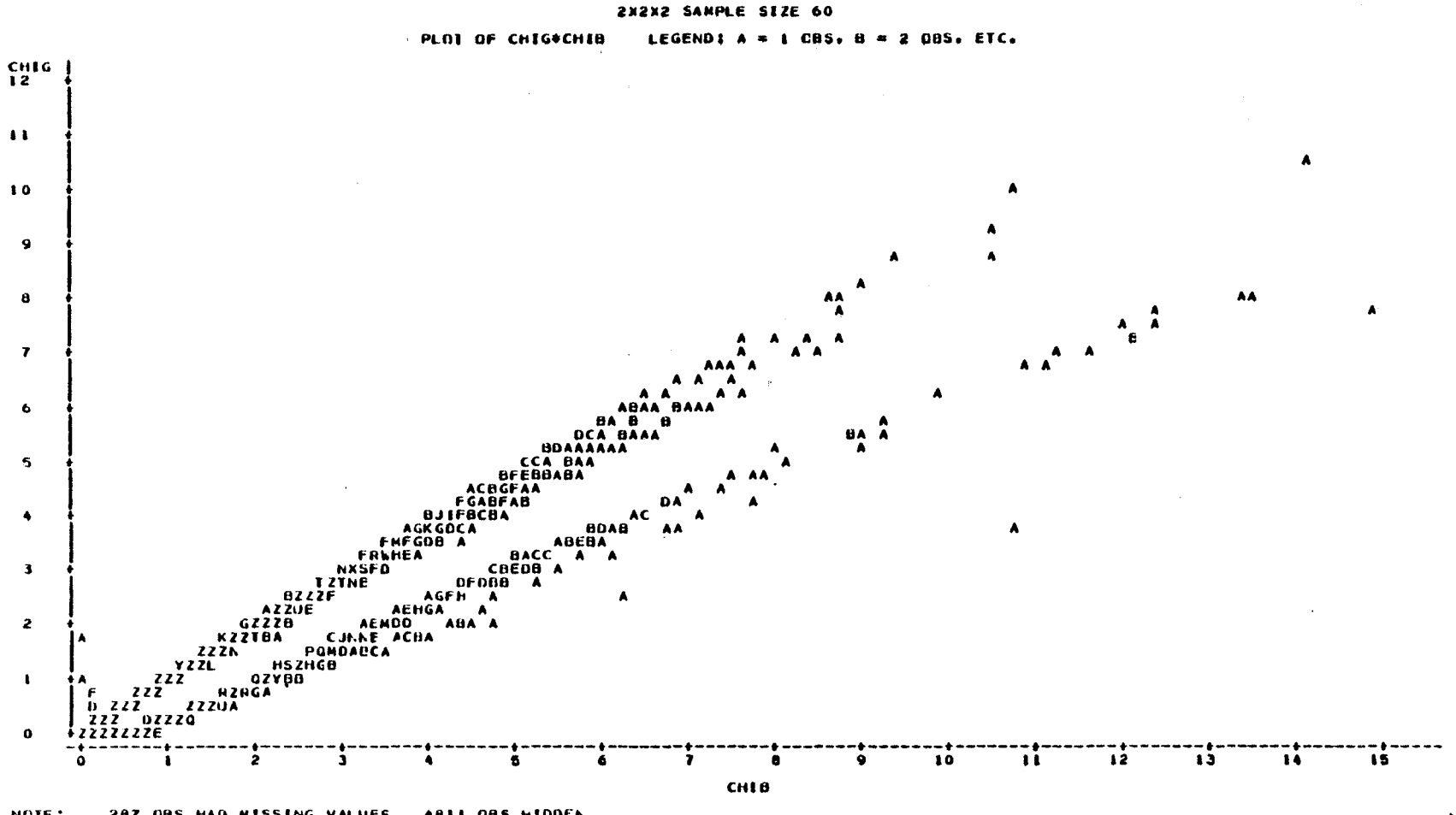

Figure 46. Scatter Diagram indicating two lines: Bartlett vs. Goodman 1

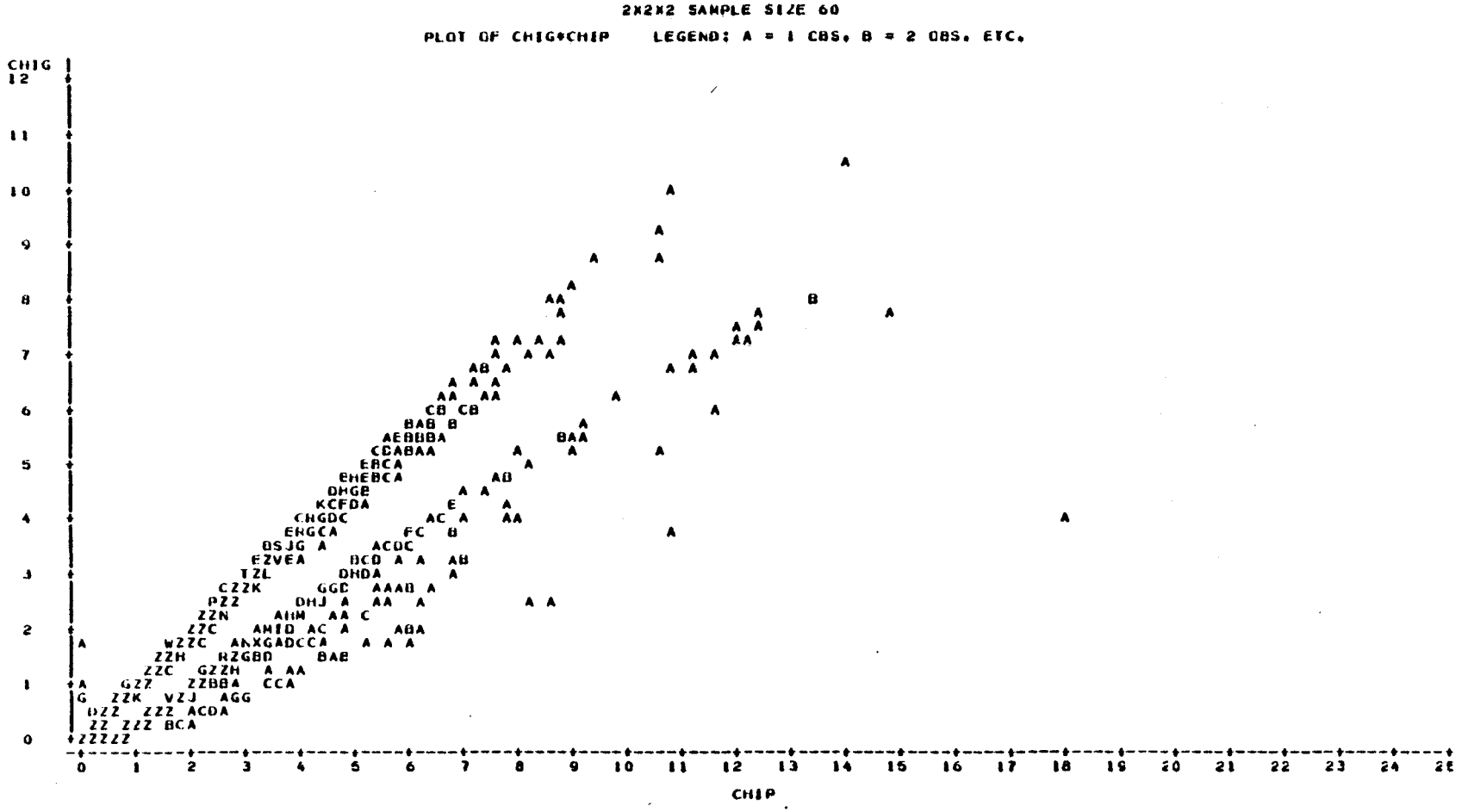

Figure 47. Scatter Diagram indicating two lines: IPF vs. Goodman 1

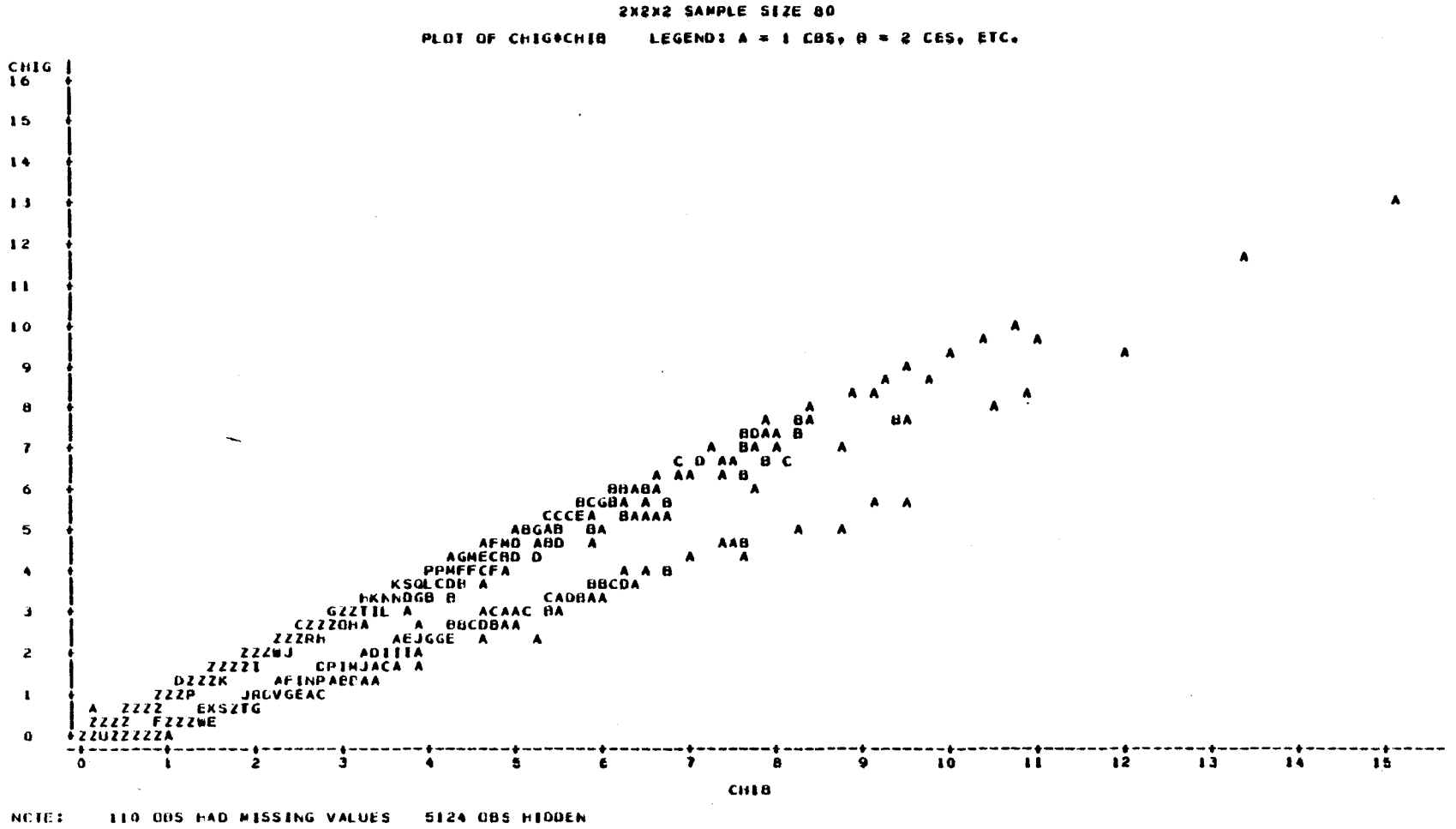

Figure 48. Scatter Diagram indicating two lines: Bartlett vs. Goodman 1

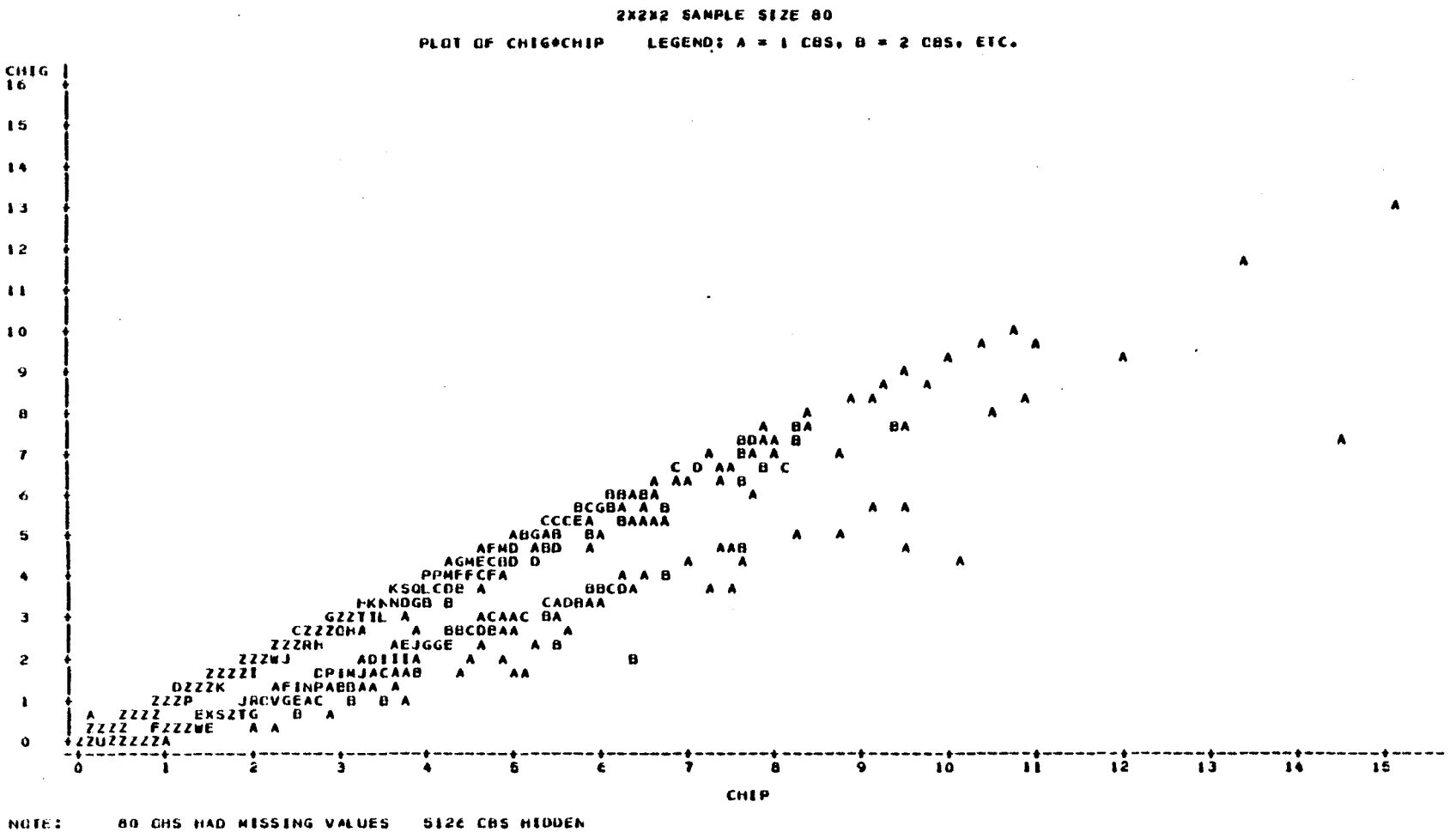

Figure 49. Scatter Diagram indicating two lines: IPF vs. Goodman 1

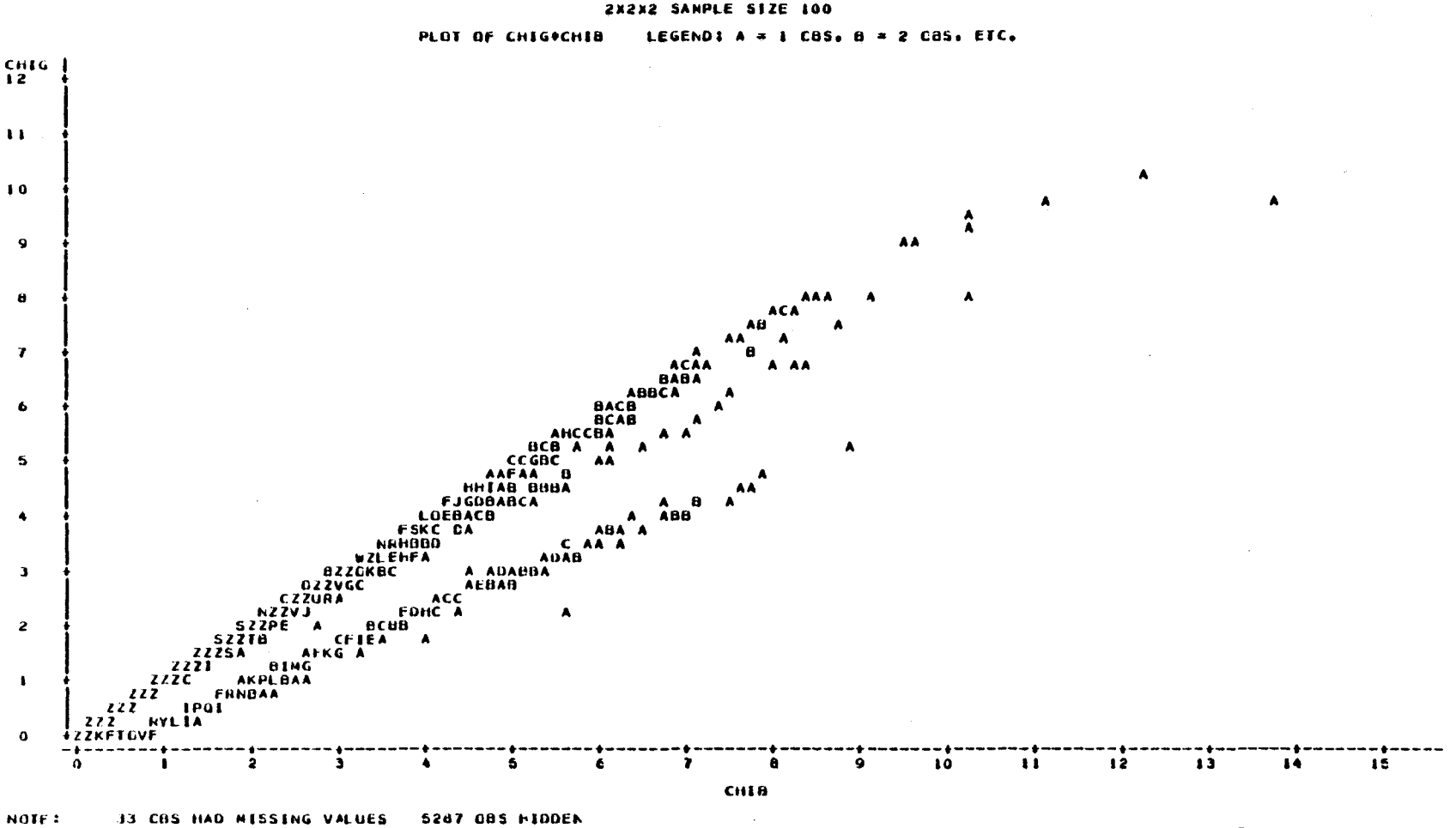

Figure 50. Scatter Diagram indicating two lines: Bartlett vs. Goodman 1

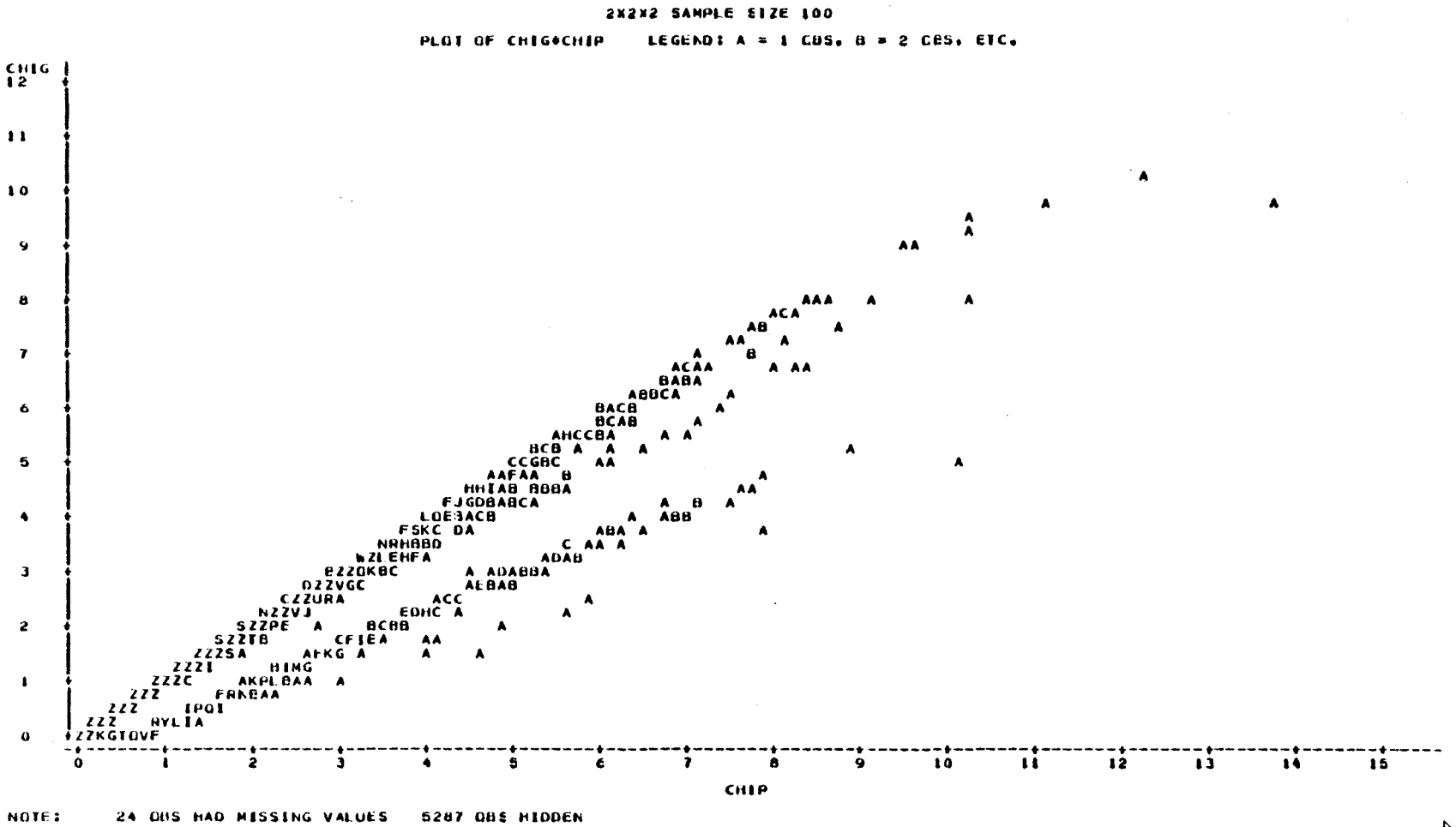

Figure 51. Scatter Diagram indicating two lines: IPF vs. Goodman 1

APPENDIX C

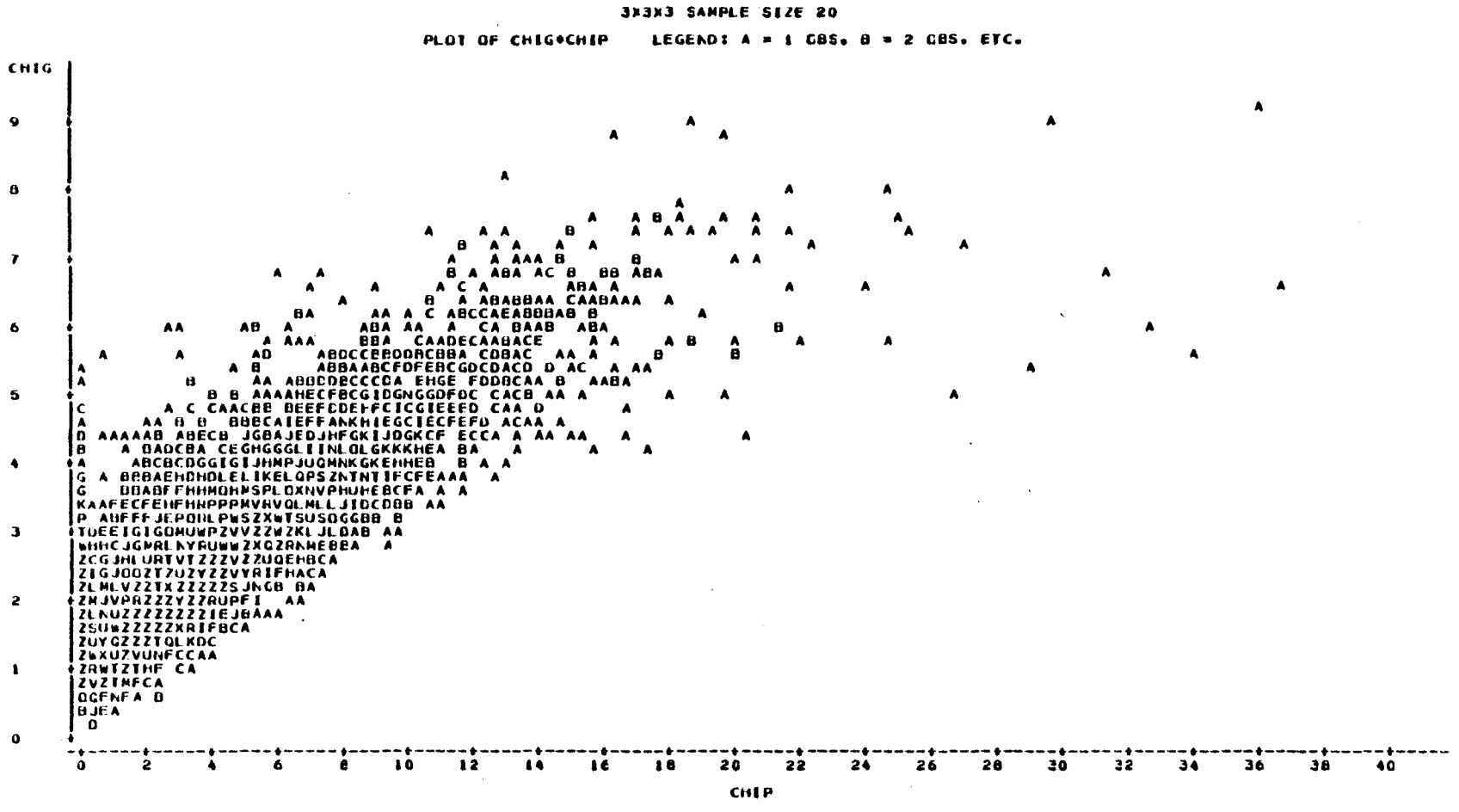

Figure 52. Scatter Diagram indicating one line: IPF vs. Goodman 2

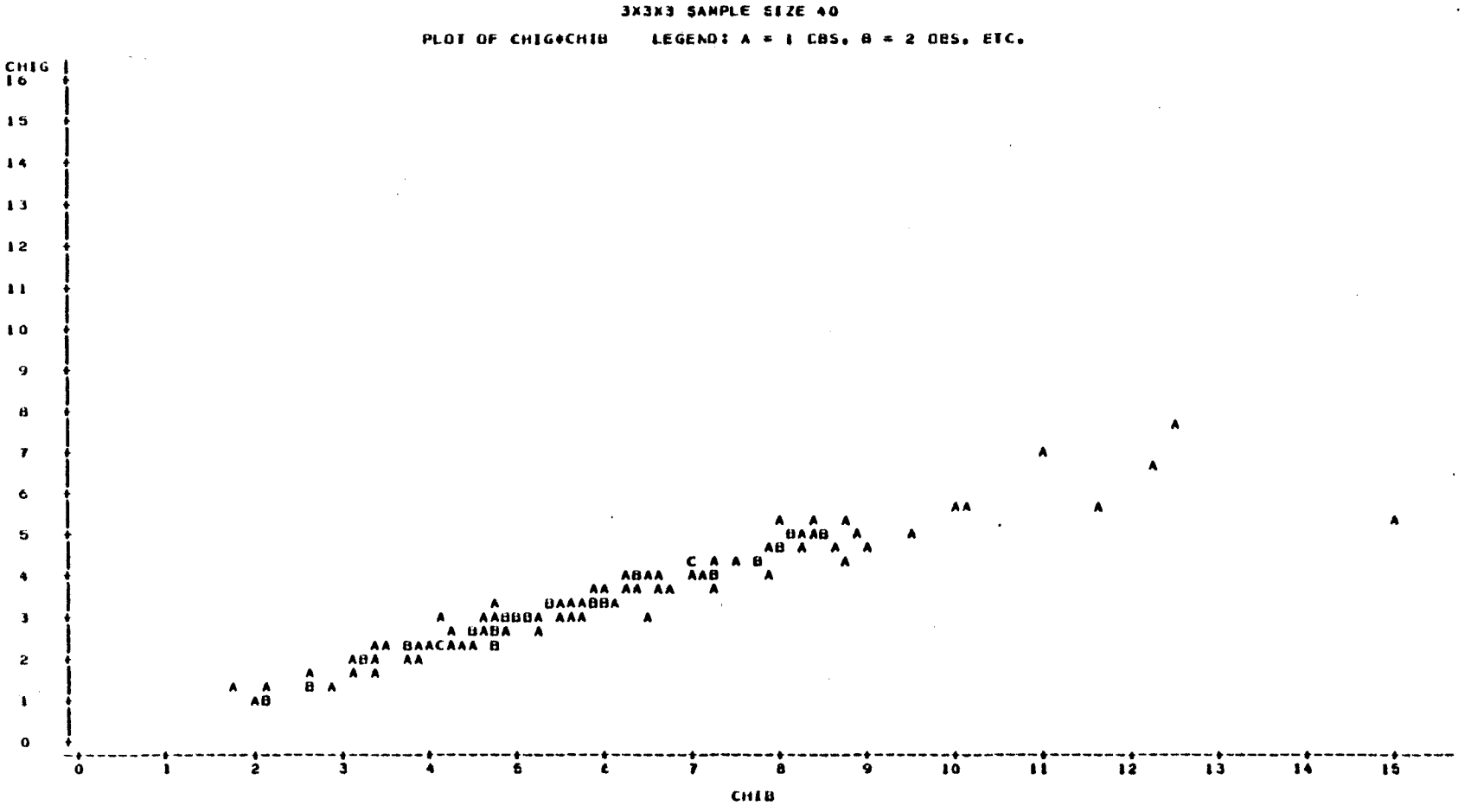

Figure 53. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

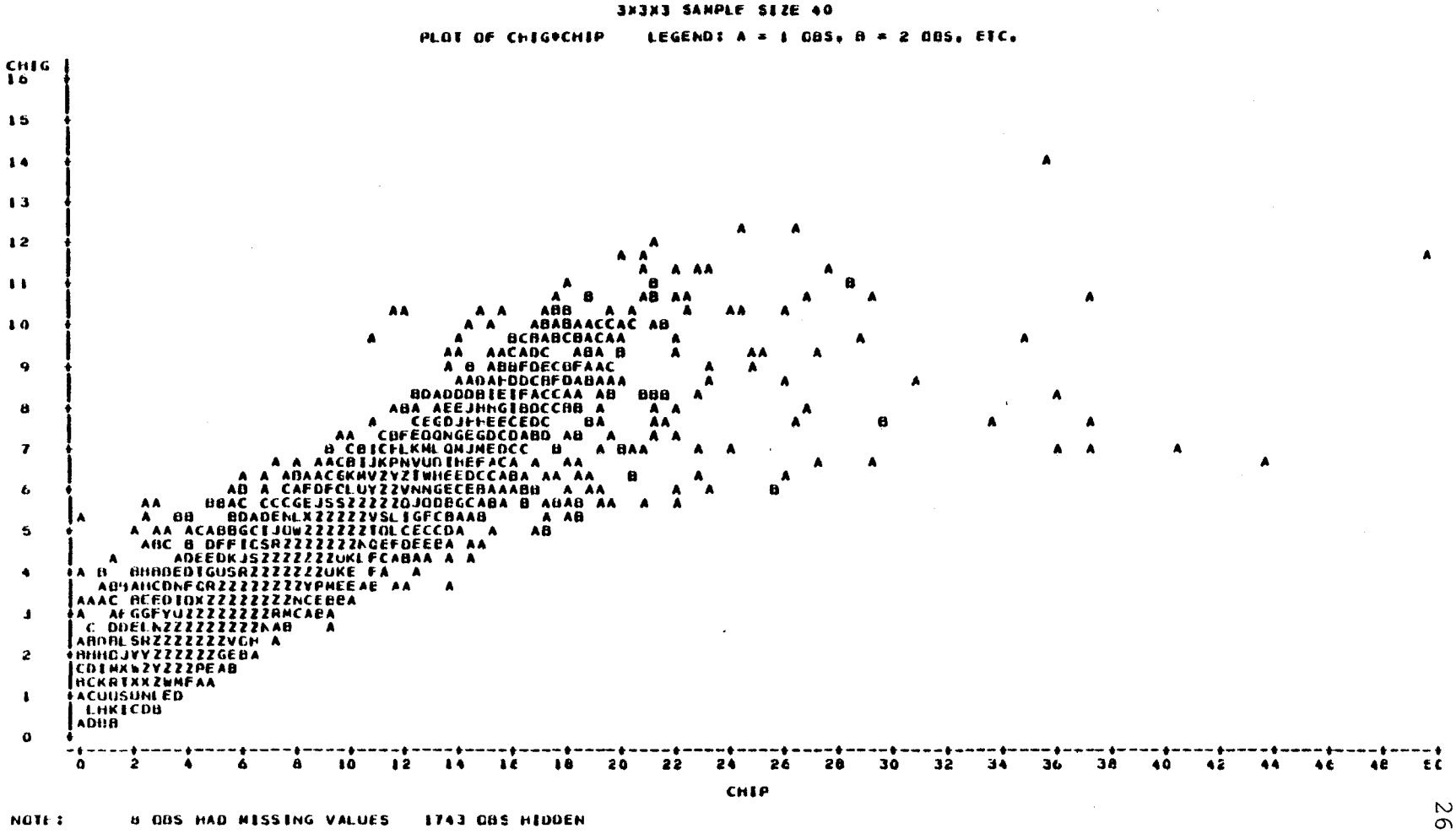

Figure 54. Scatter Diagram indicating one line: IPF vs. Goodman 2

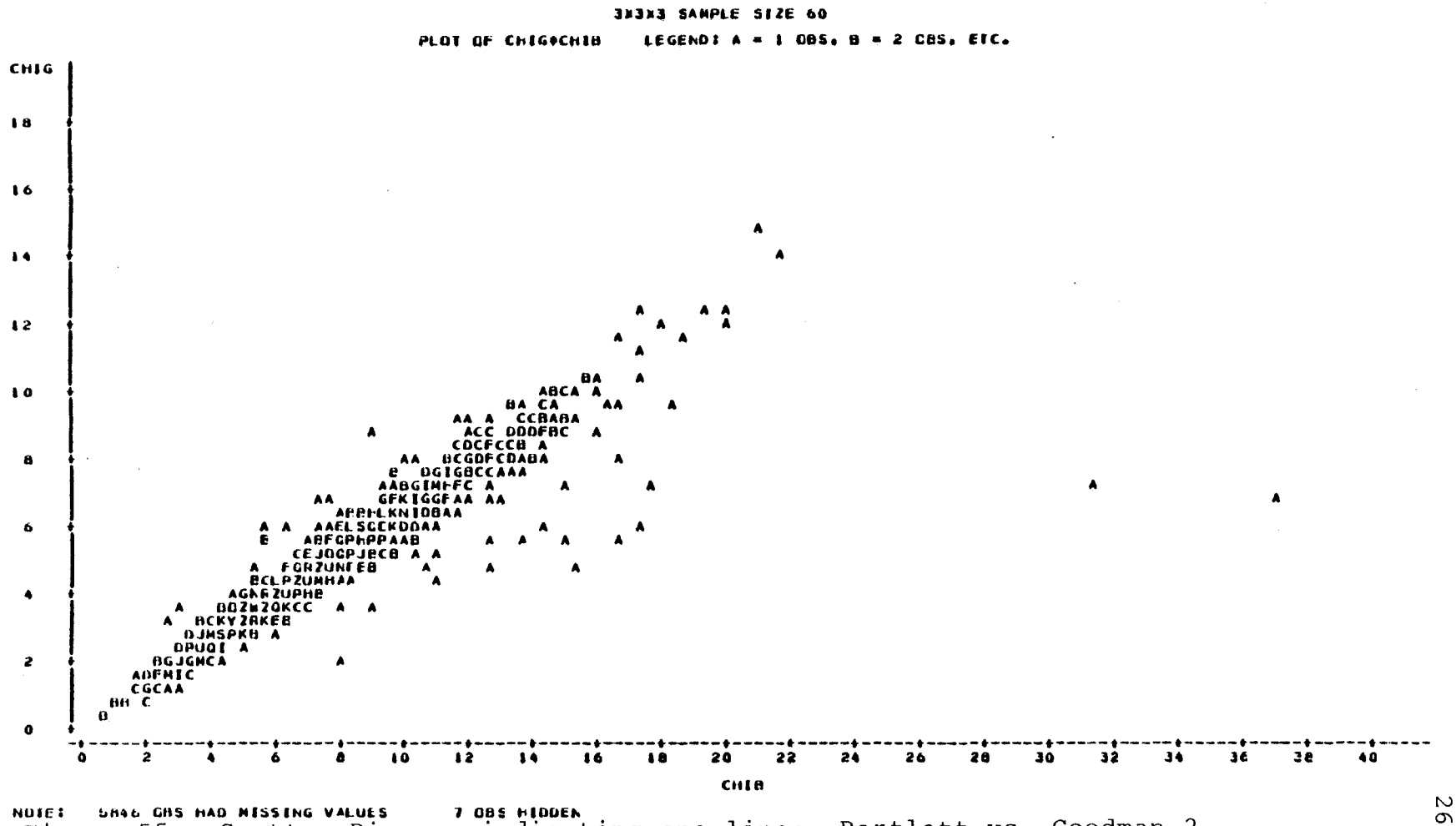

Figure 55. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

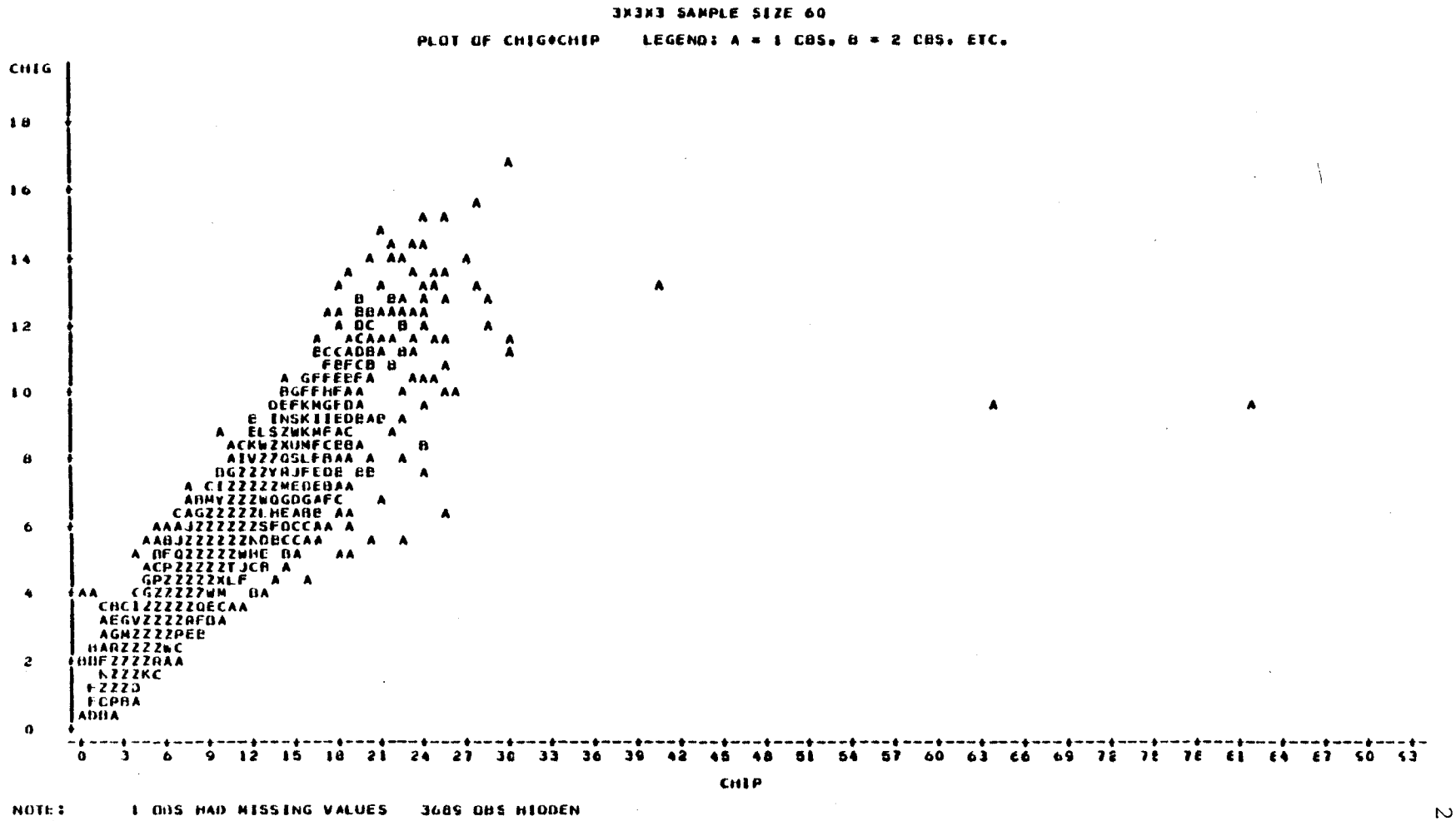

Figure 56. Scatter Diagram indicating one line: IPF vs. Goodman 2

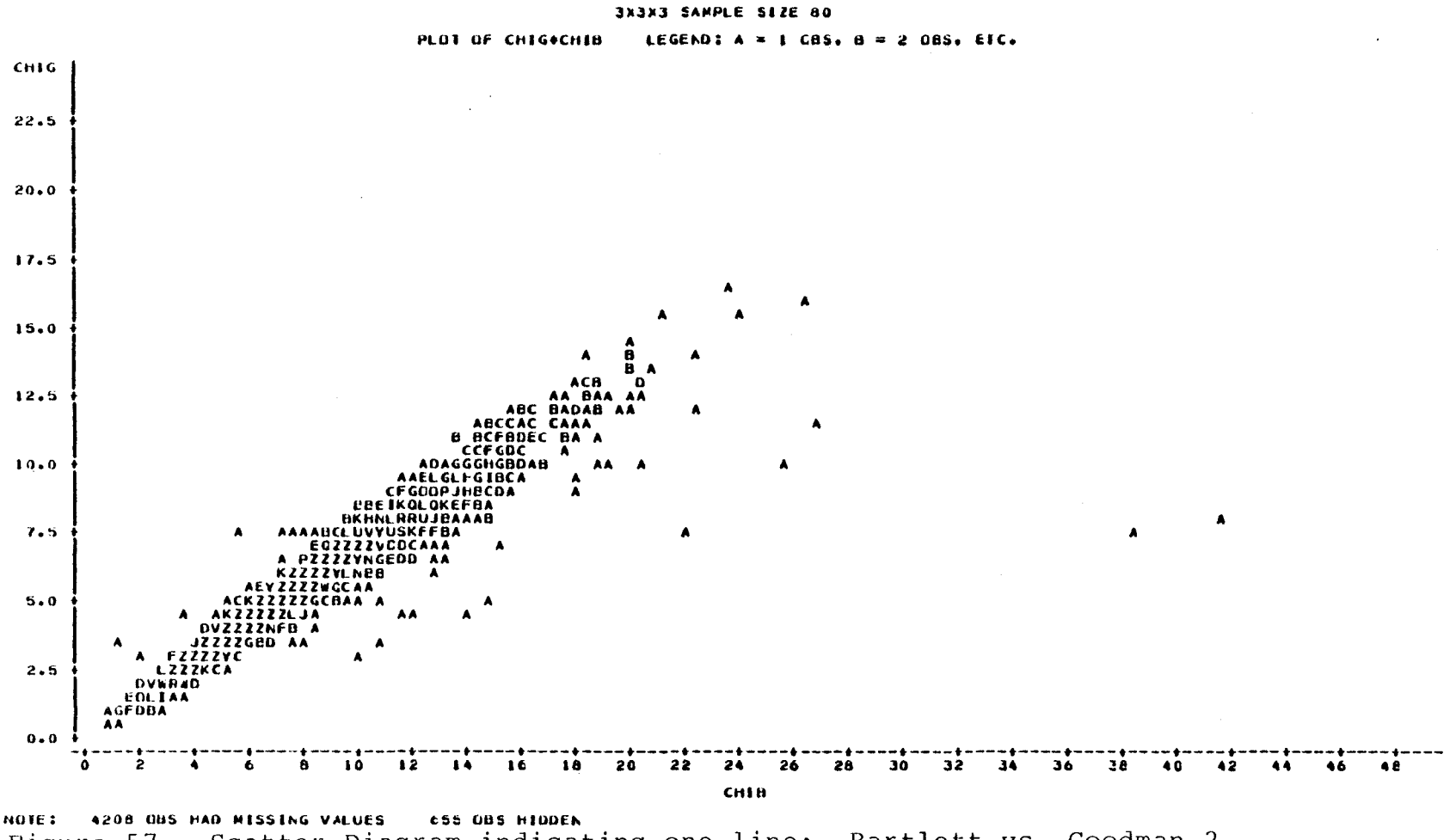

Figure 57. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

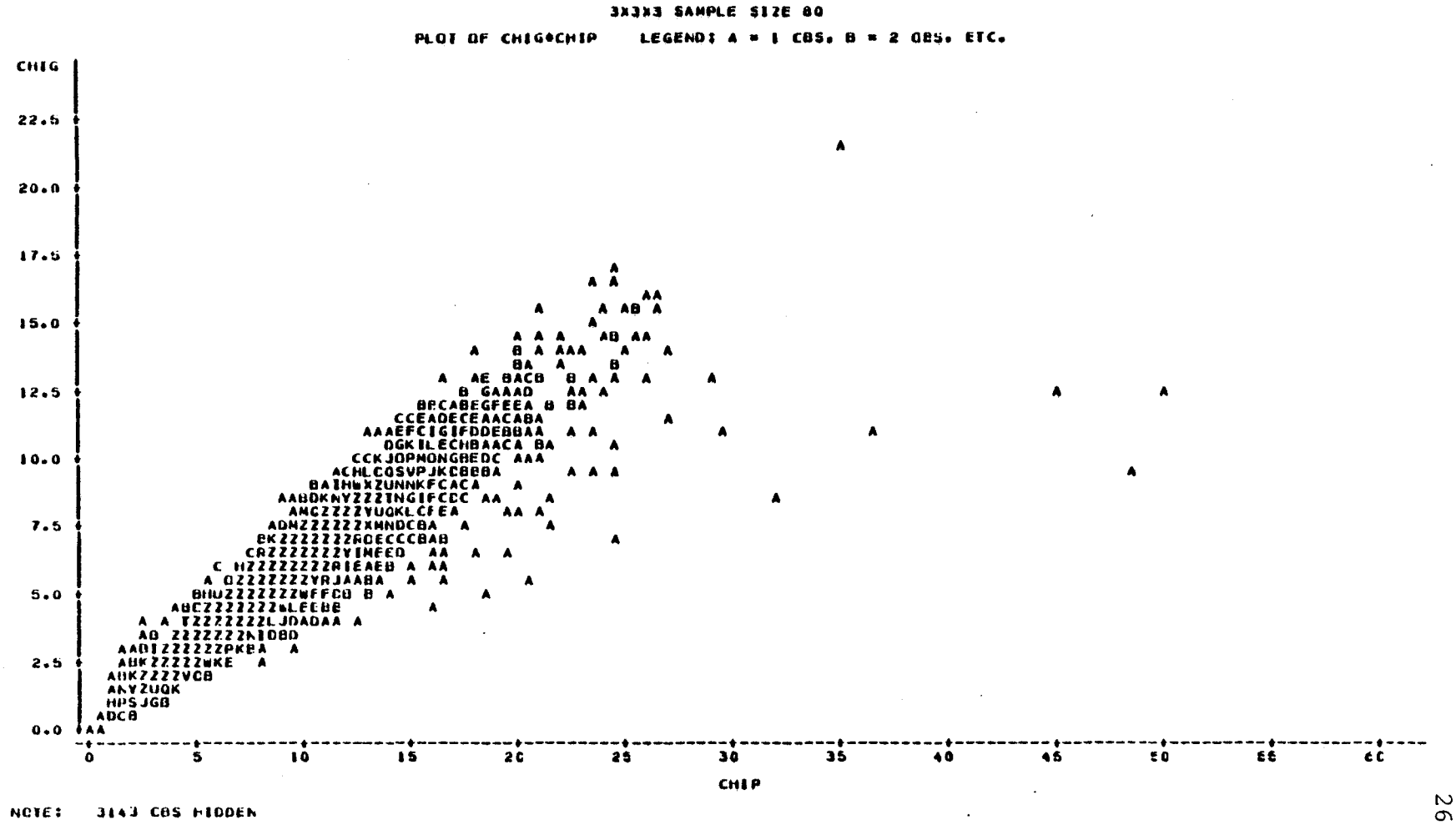

Figure 58. Scatter Diagram indicating one line: IPF vs. Goodman 2

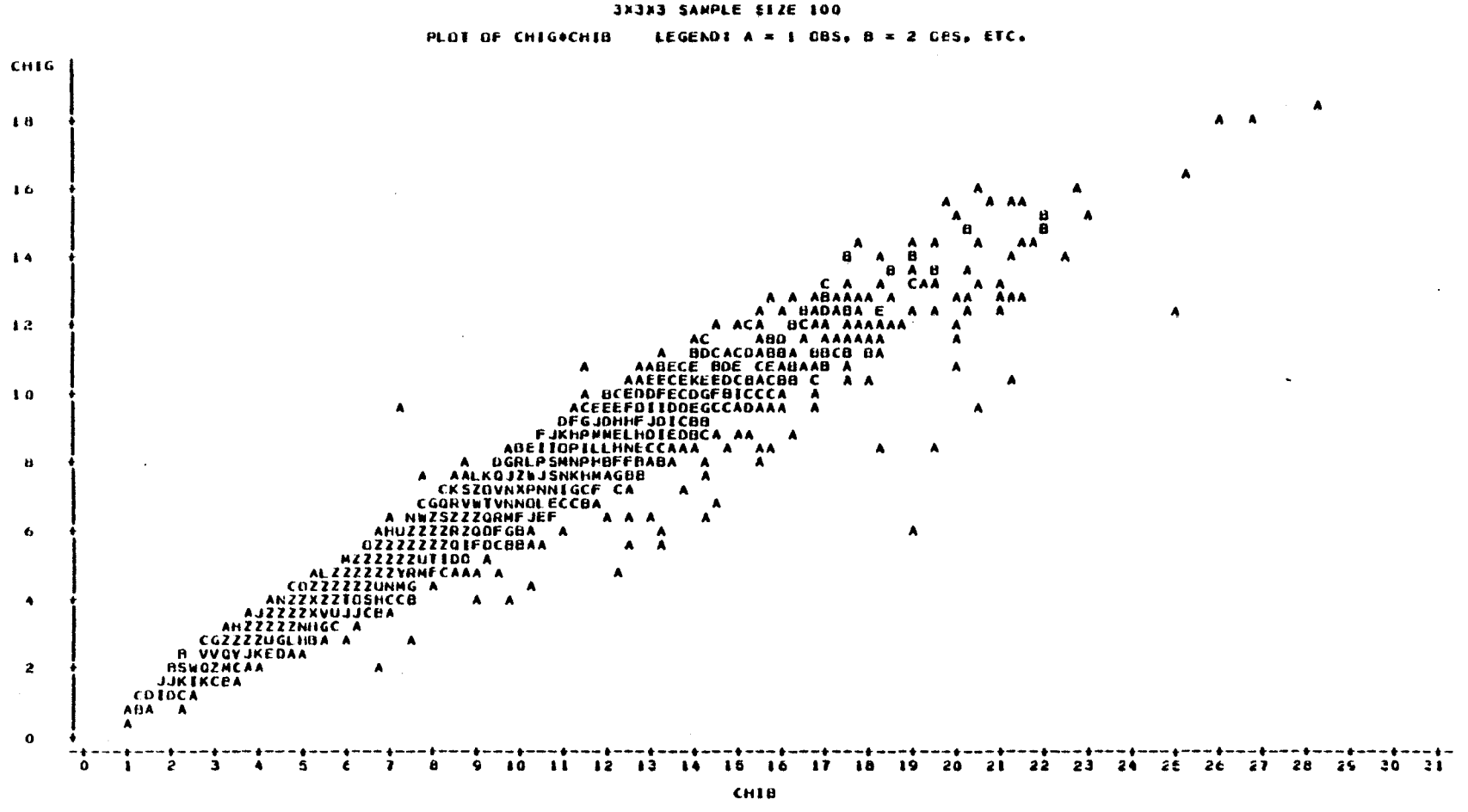

Figure 59. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

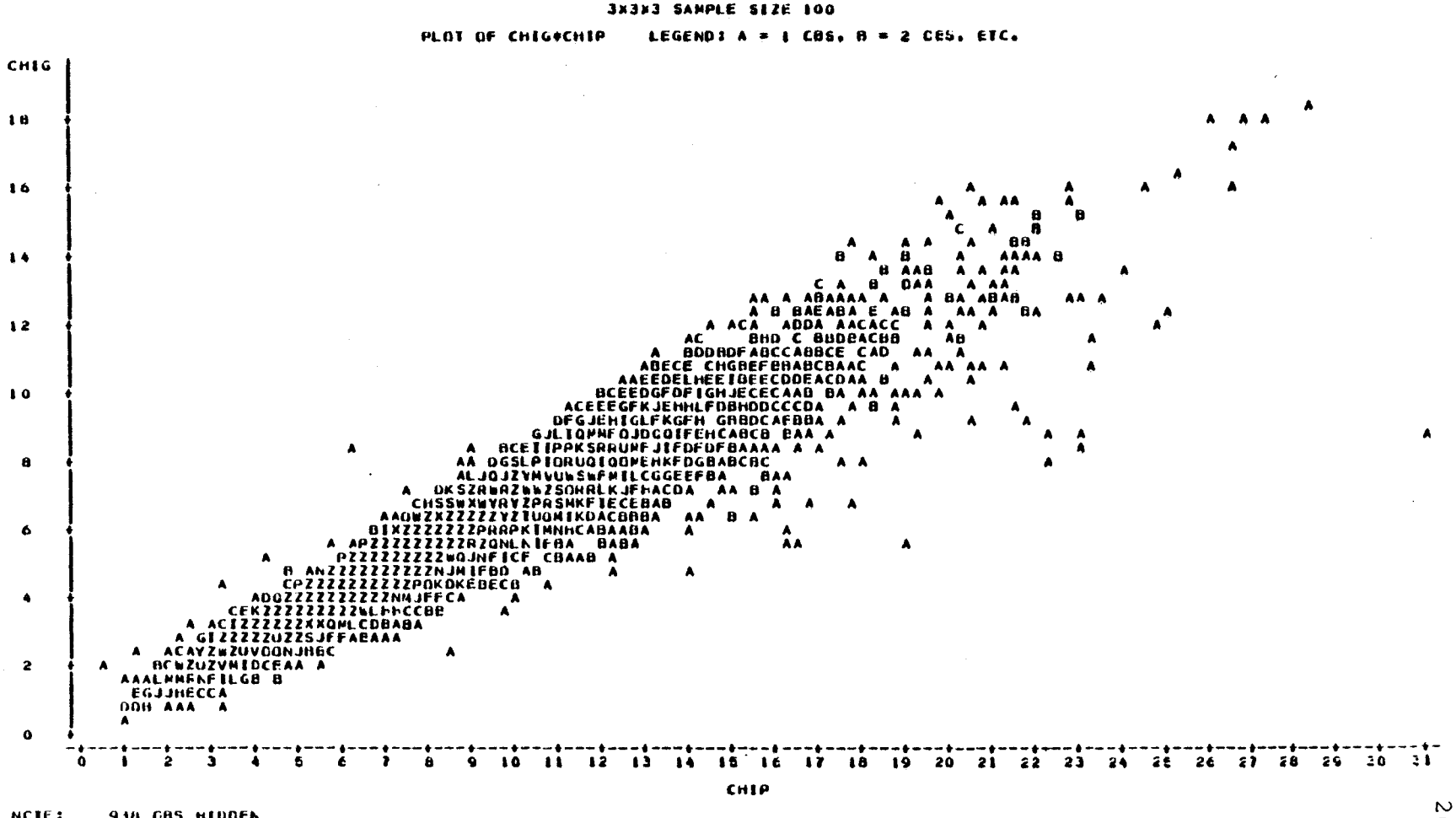

Figure 60. Scatter Diagram indicating one line: IPF vs. Goodman 2

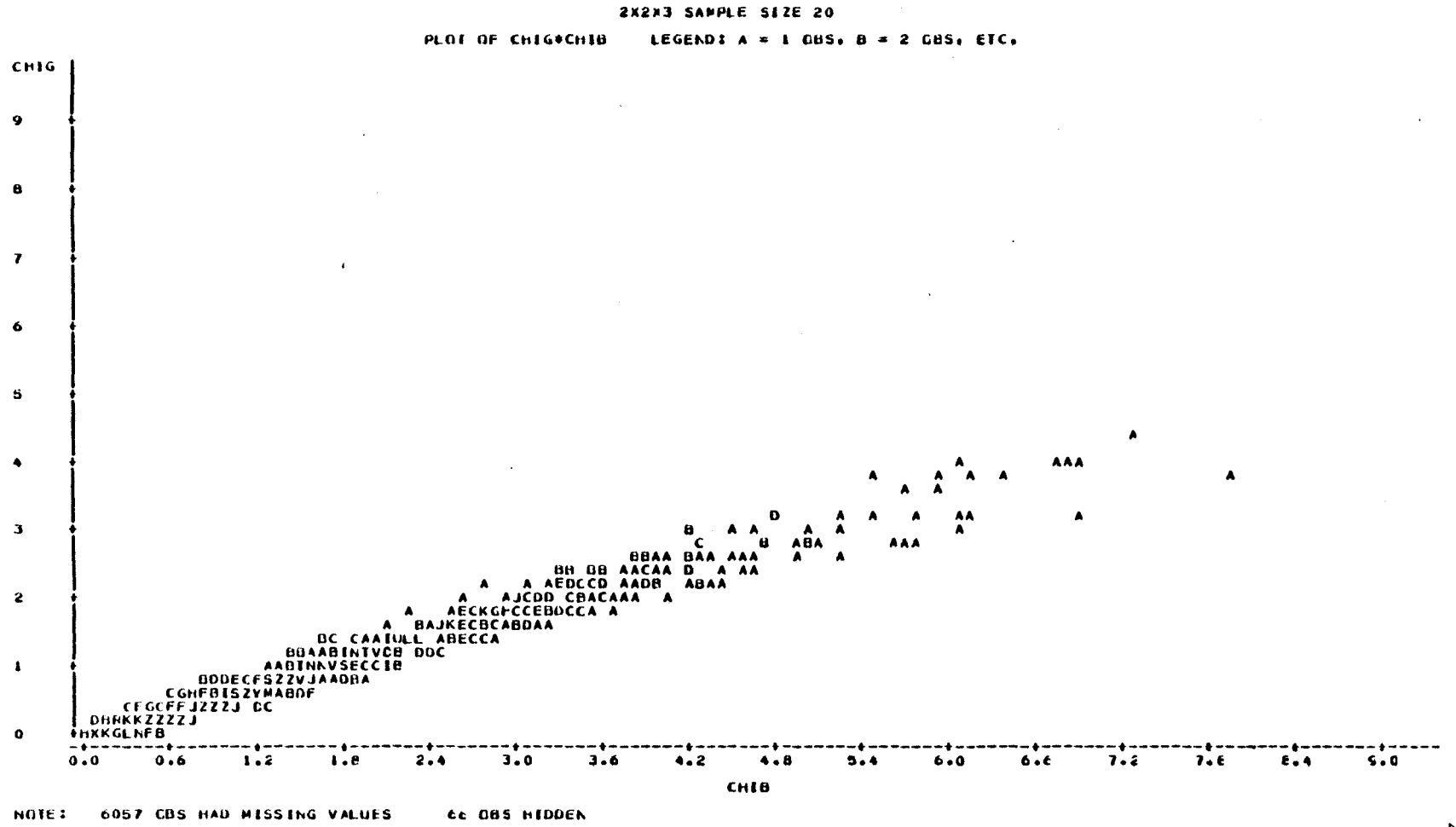

Figure 61. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

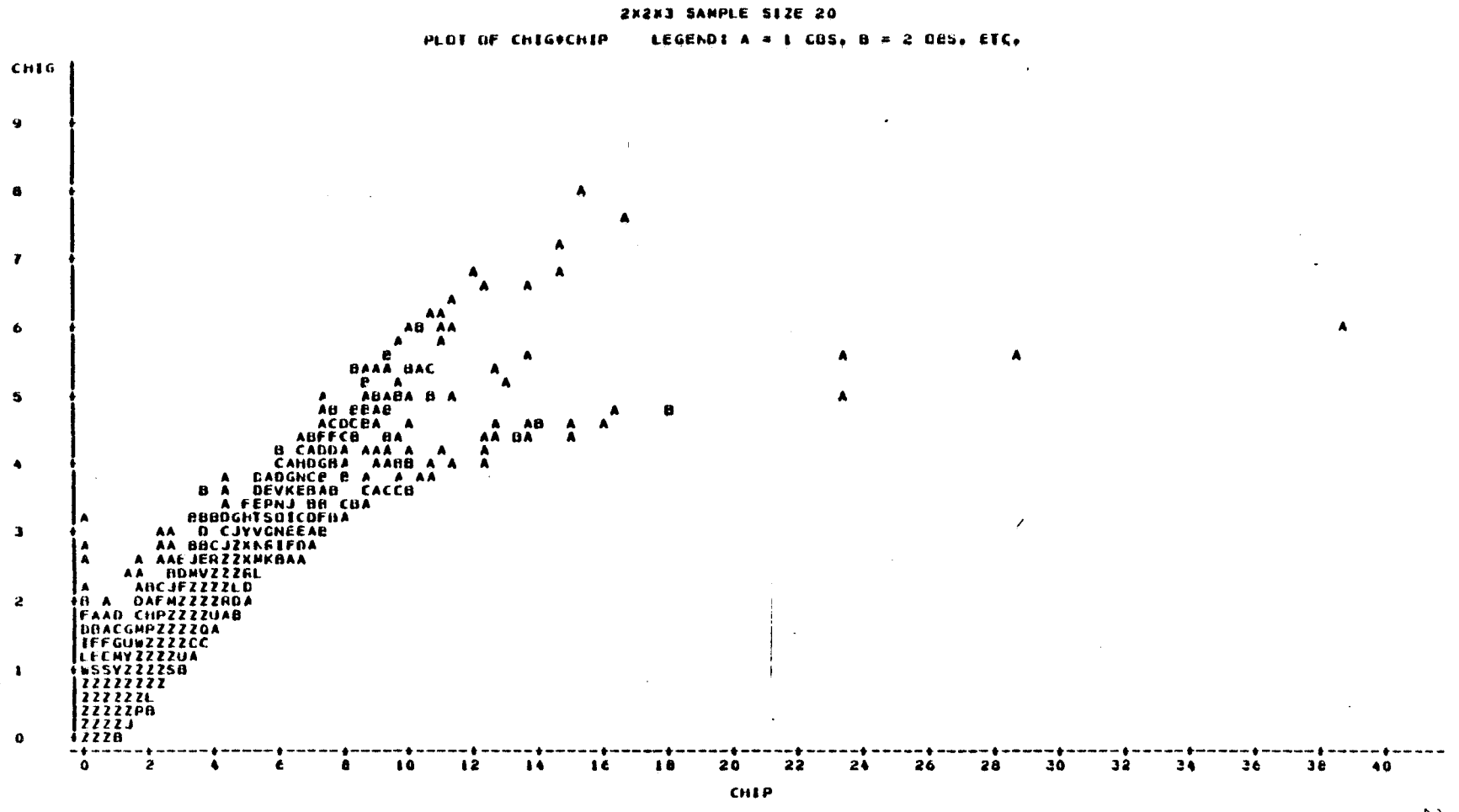

Figure 62. Scatter Diagram indicating one line: IPF vs. Goodman 2

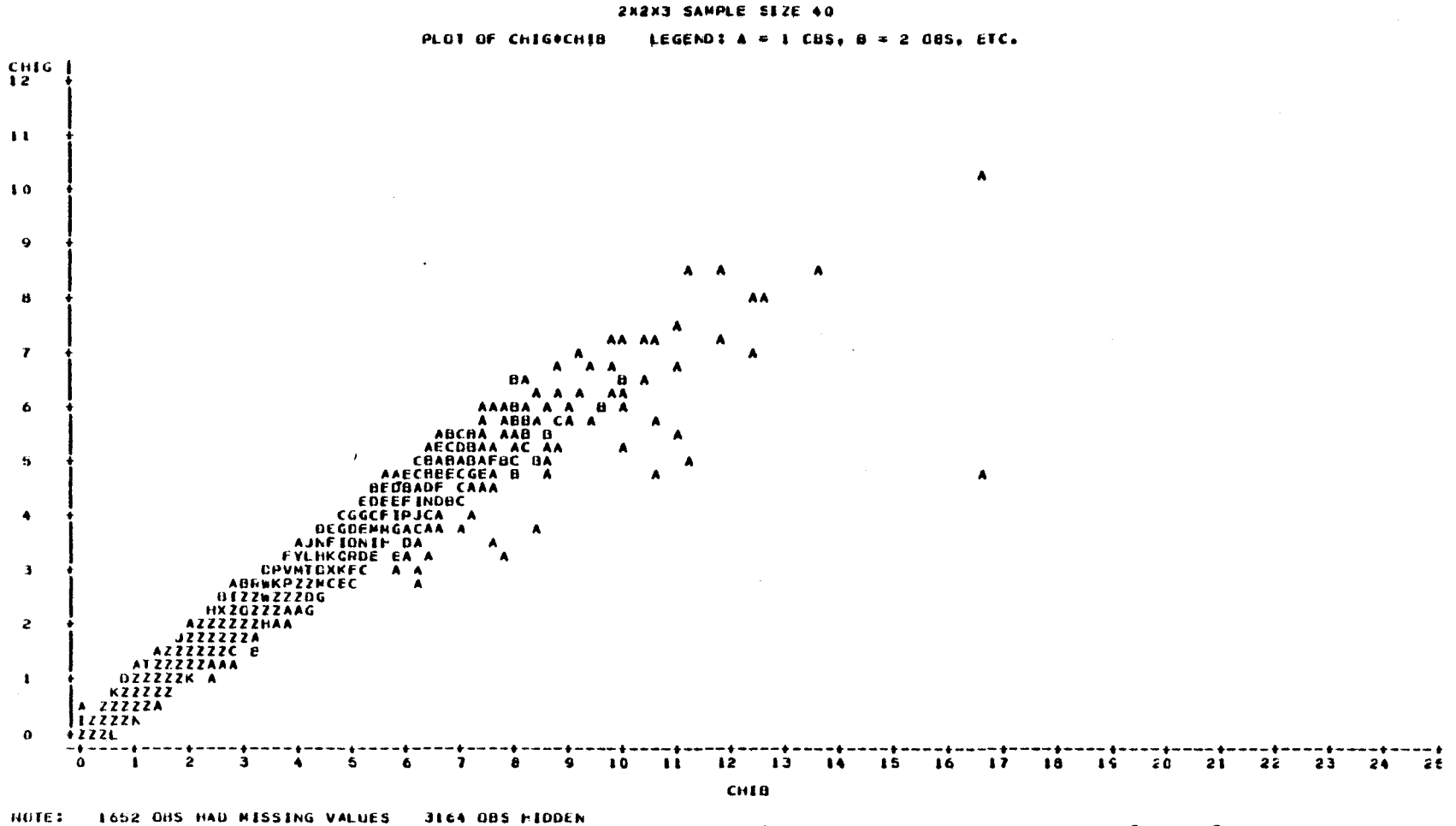

Figure 63. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

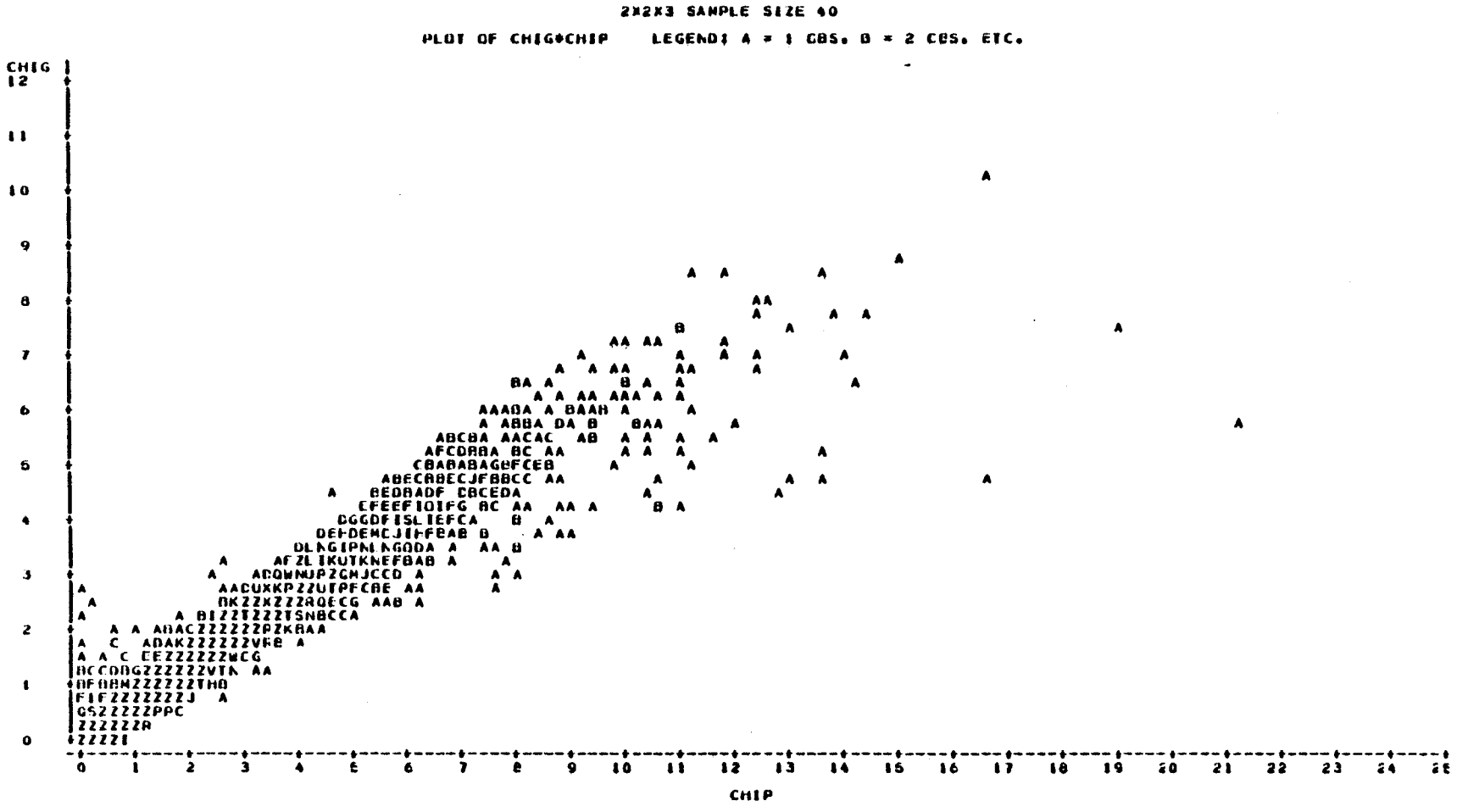

Figure 64. Scatter Diagram indicating one line: IPF vs. Goodman 2

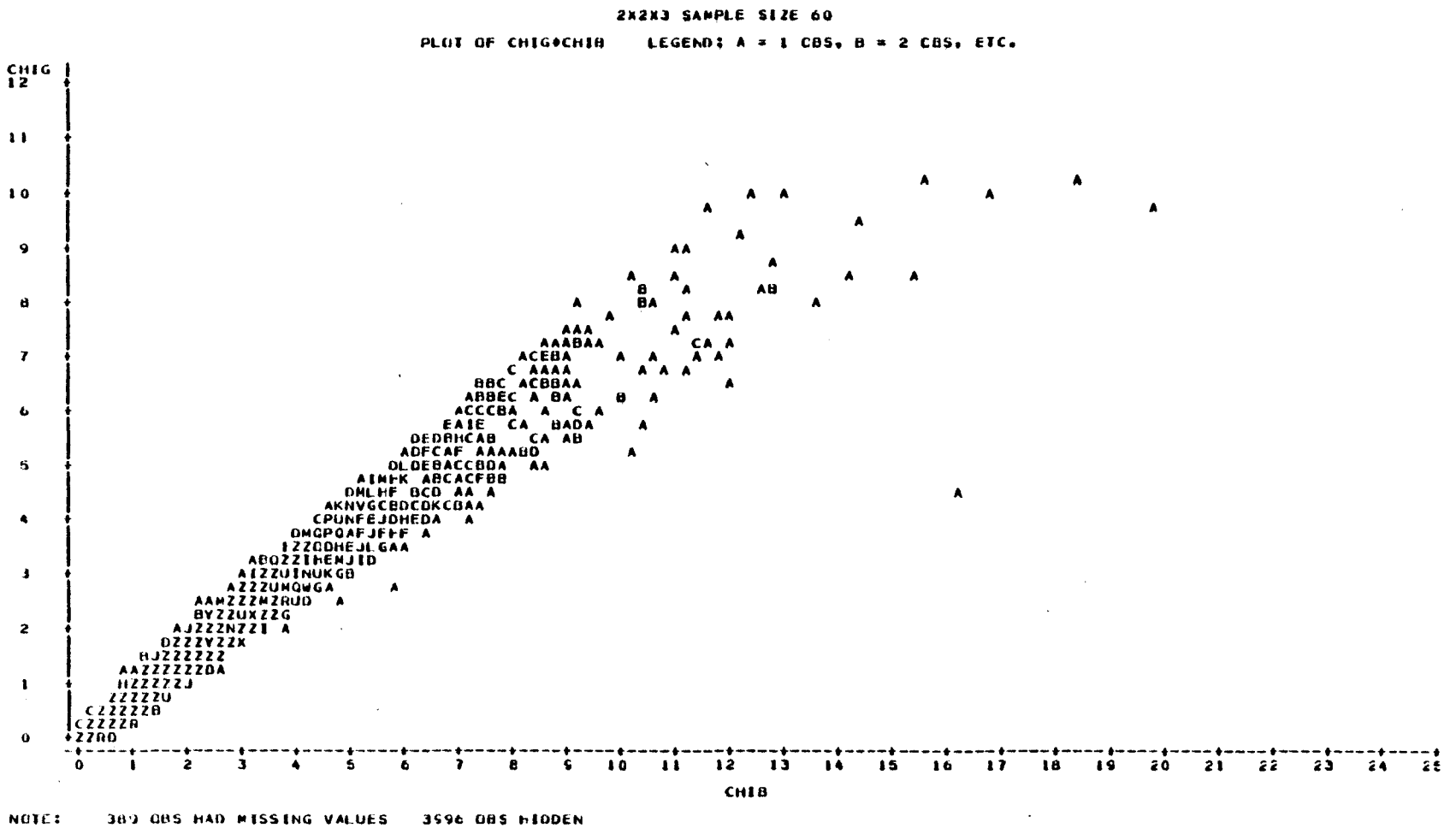

Figure 65. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

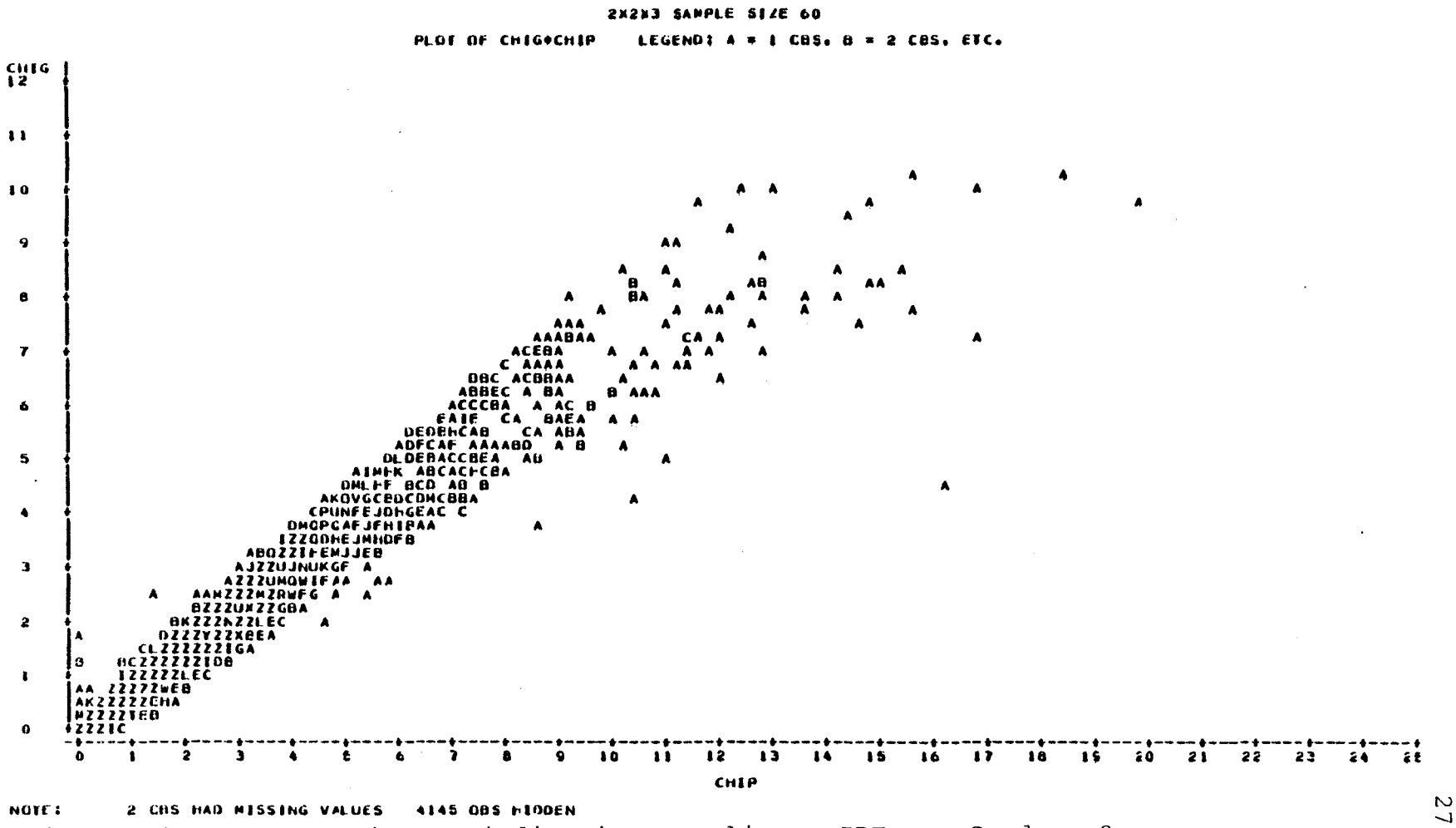

Figure 66. Scatter Diagram indicating one line: IPF vs. Goodman 2

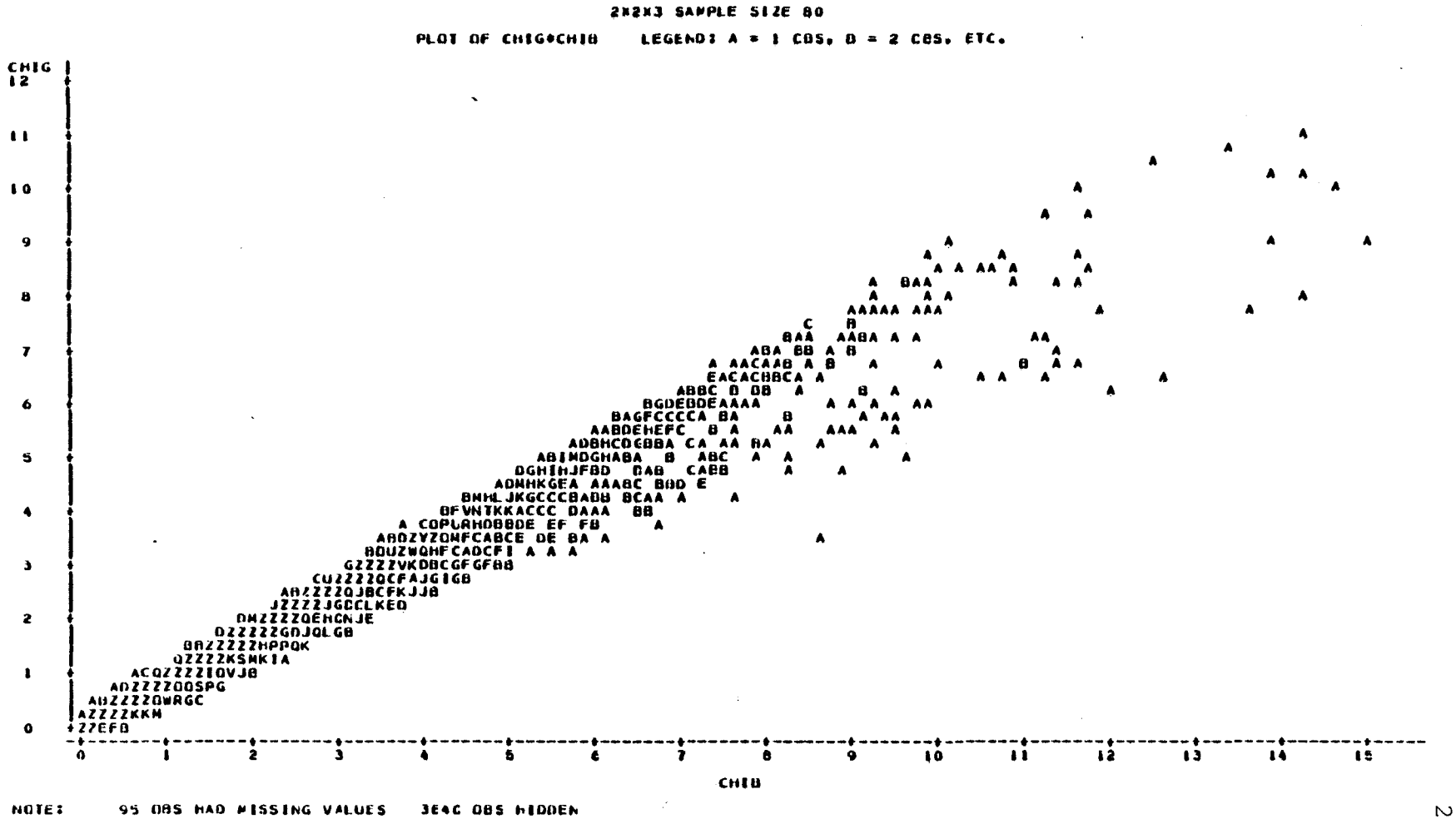

Figure 67. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

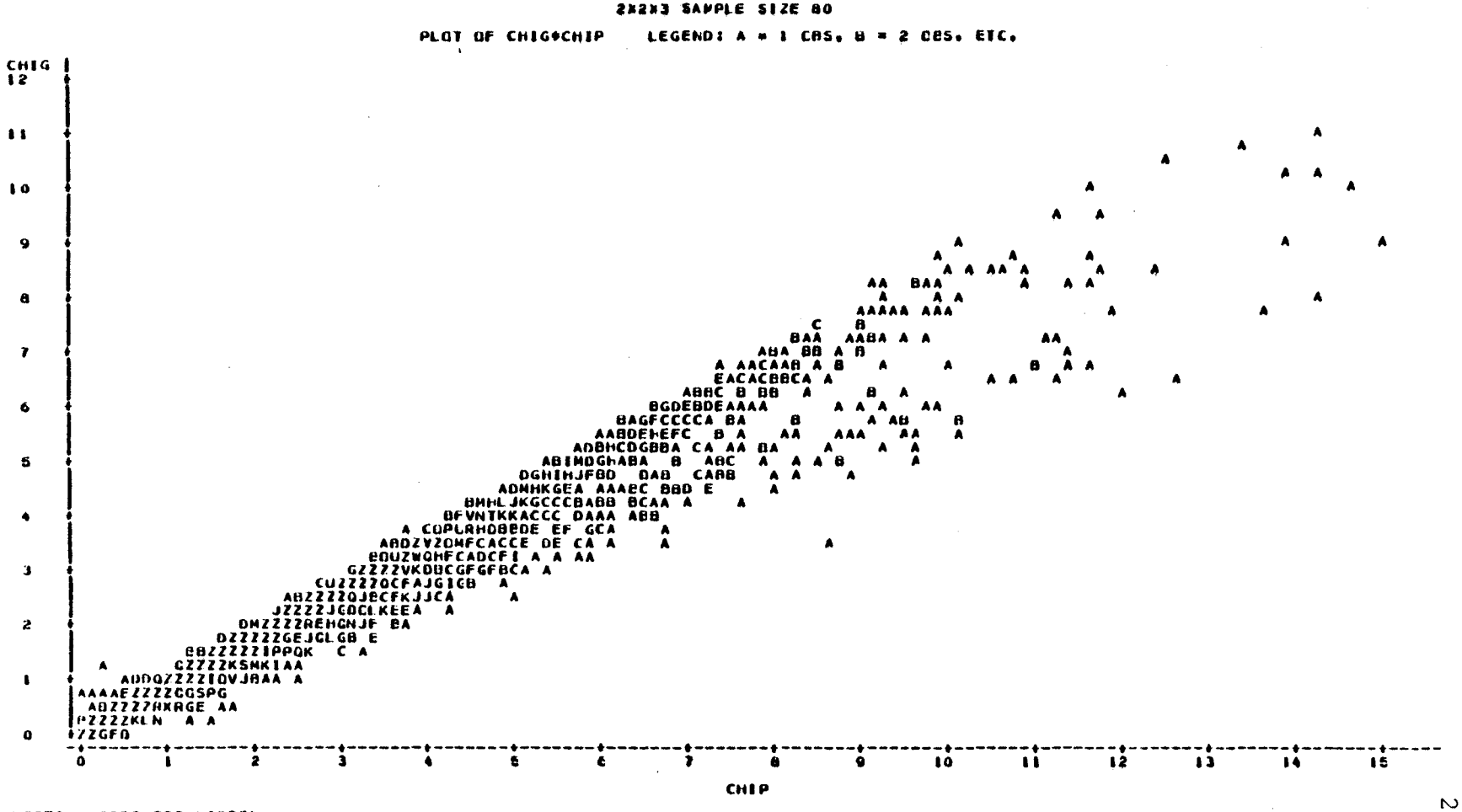

Figure 68. Scatter Diagram indicating one line: IPF vs. Goodman 2

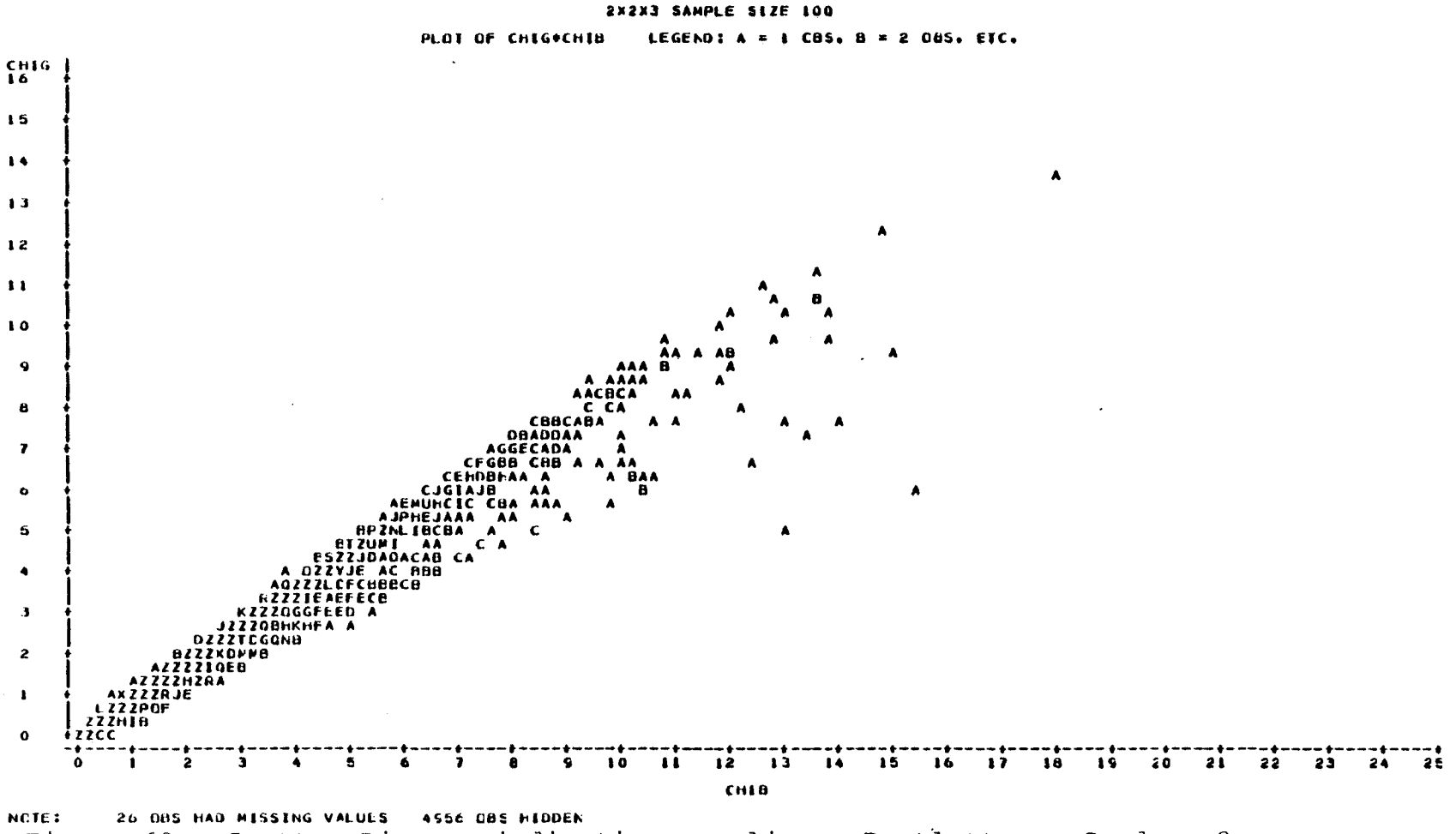

Figure 69. Scatter Diagram indicating one line; Bartlett vs. Goodman 2

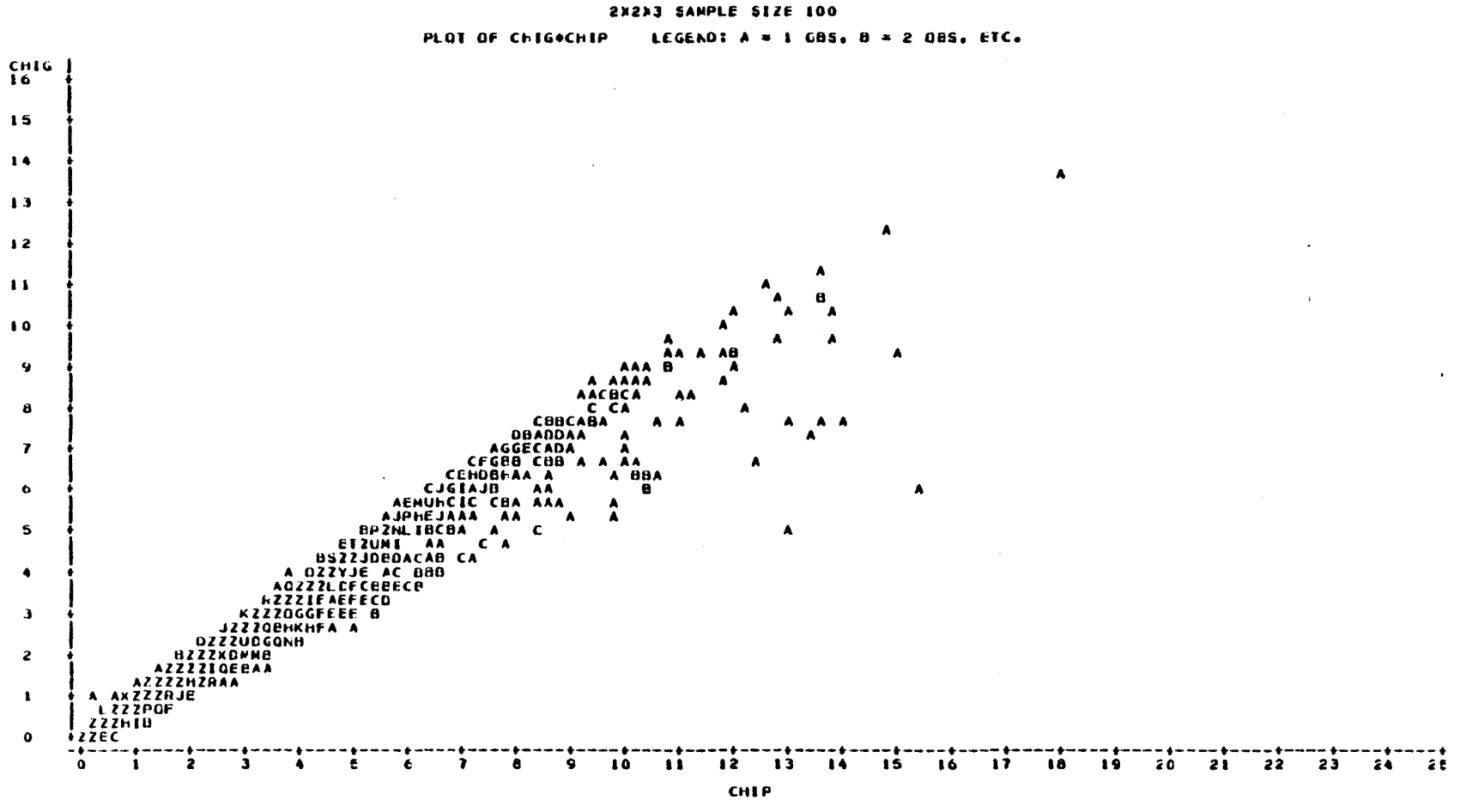

Figure 70. Scatter Diagram indicating one line: IPF vs. Goodman 2

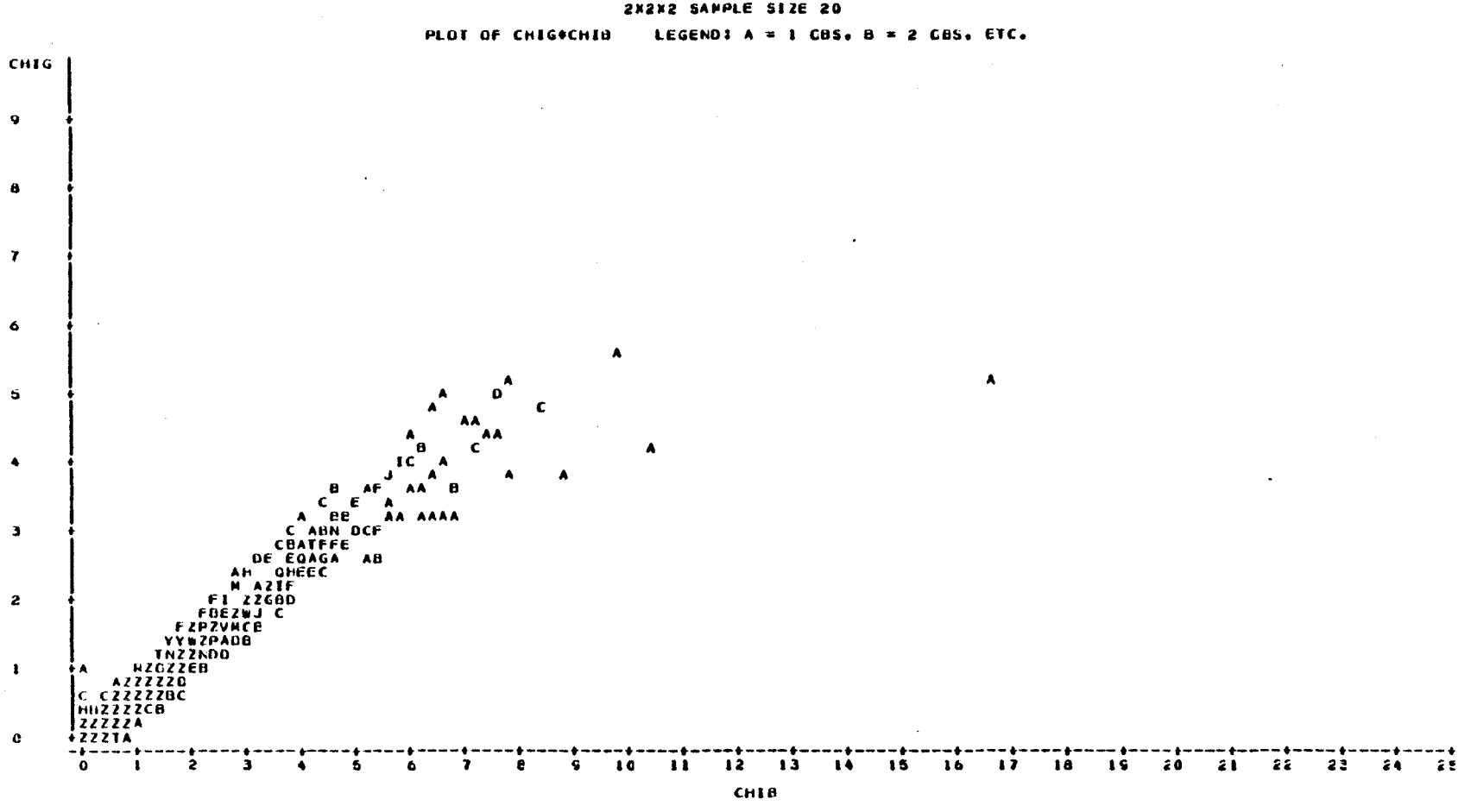

Figure 71. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

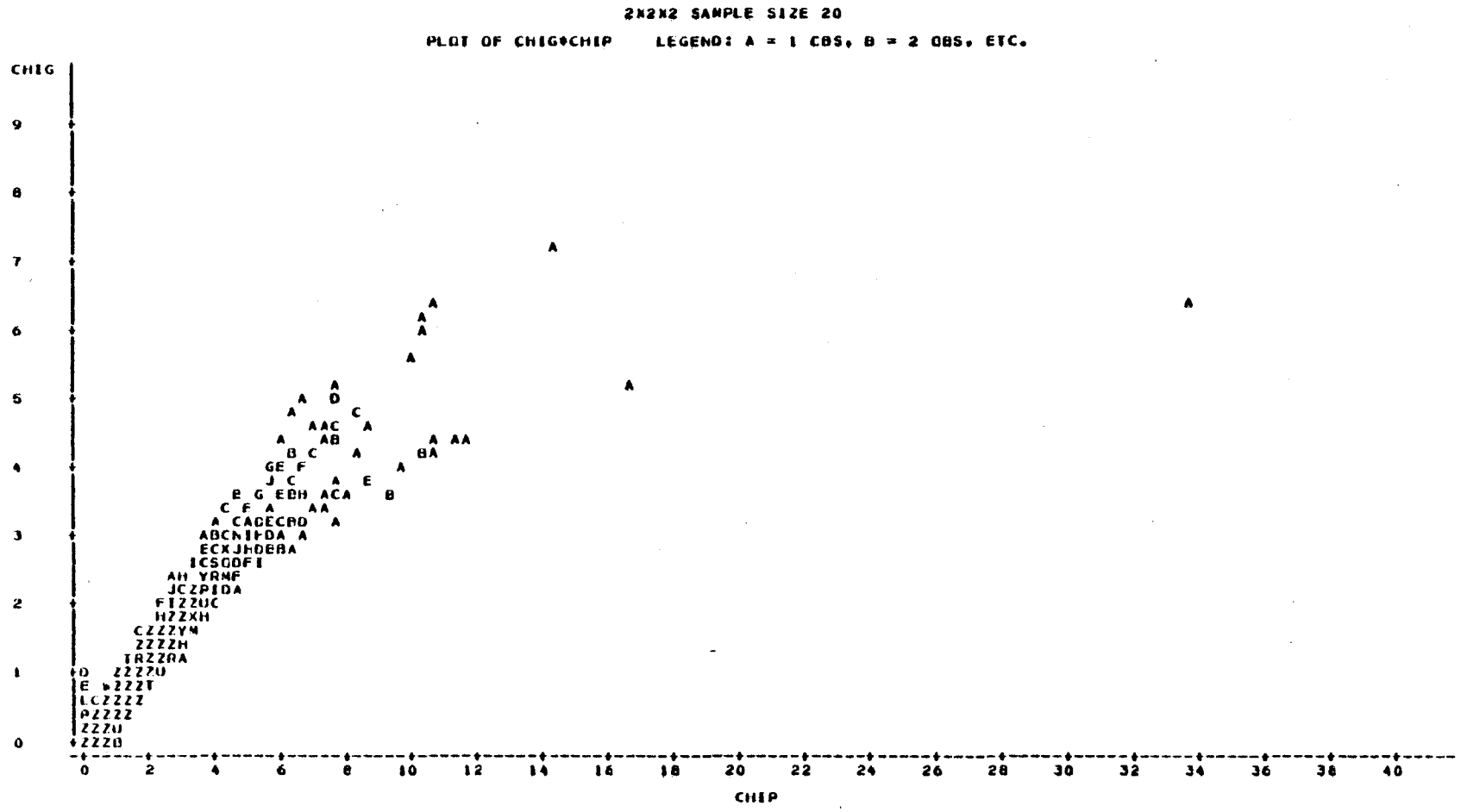

Figure 72. Scatter Diagram indicating one line: IPF vs. Goodman 2

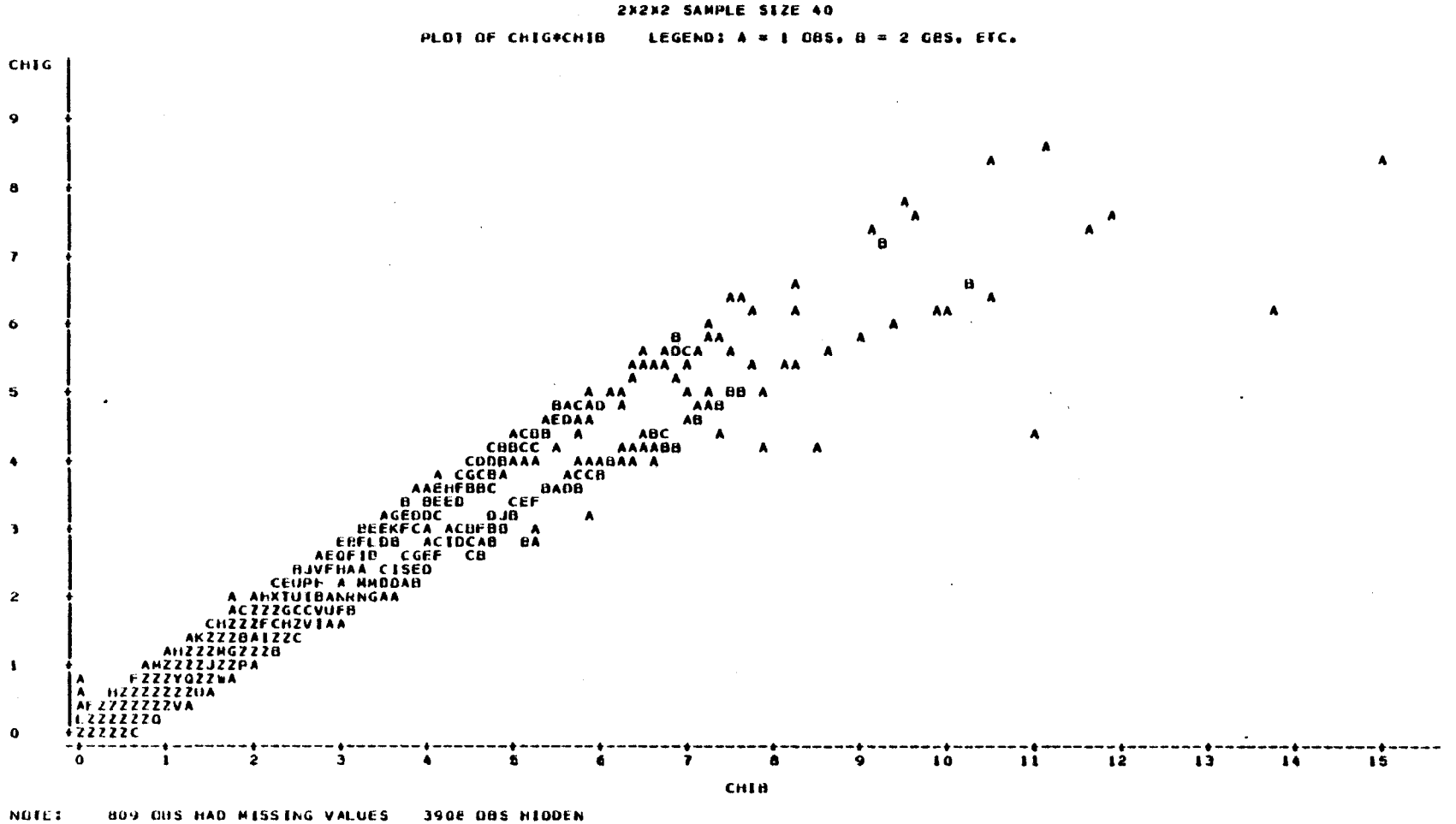

Figure 73. Scatter Diagram indicating one line: Bartlett vs. Goodman 2

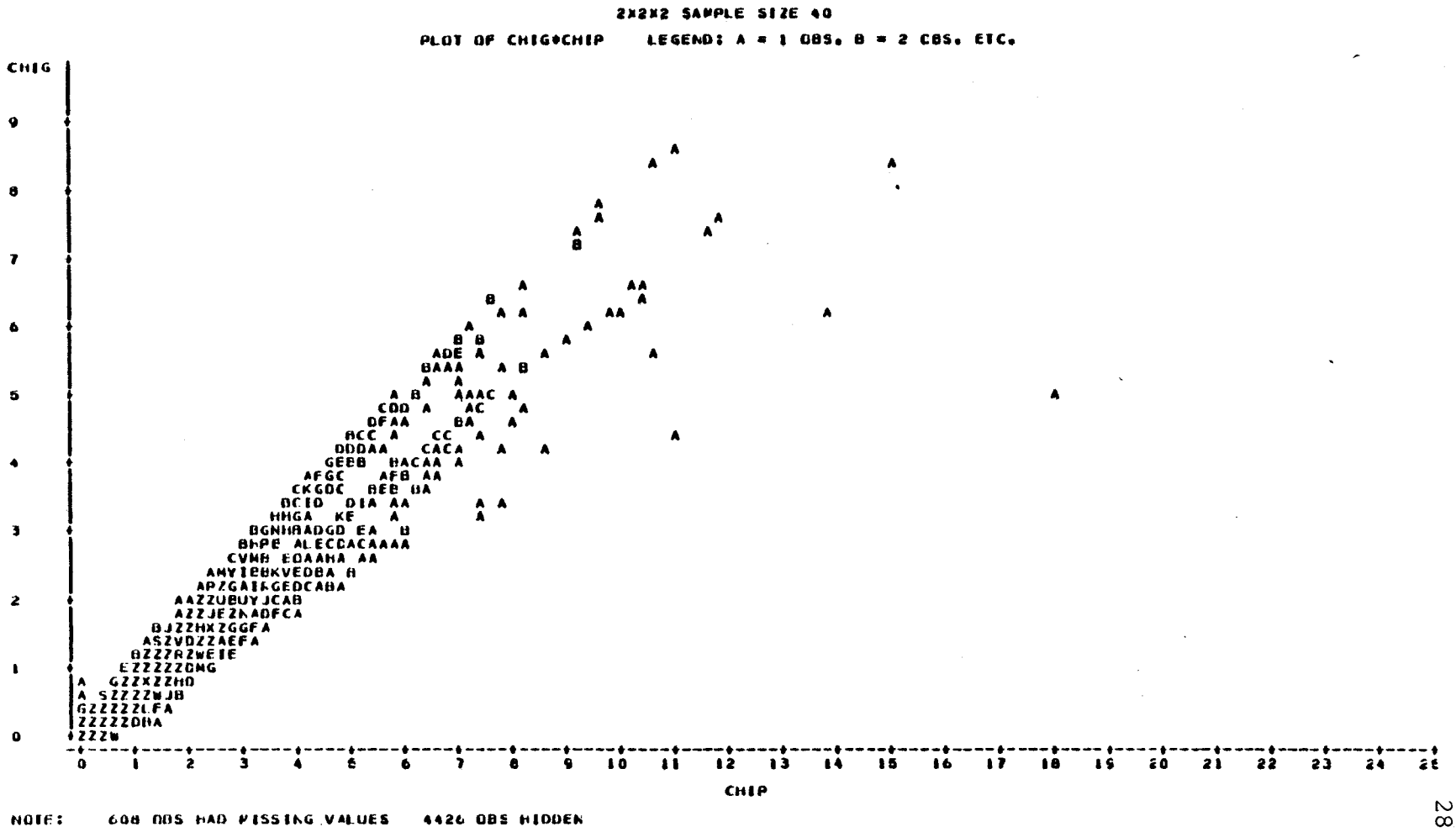

Figure 74. Scatter Diagram indicating one line: IPF vs. Goodman 2

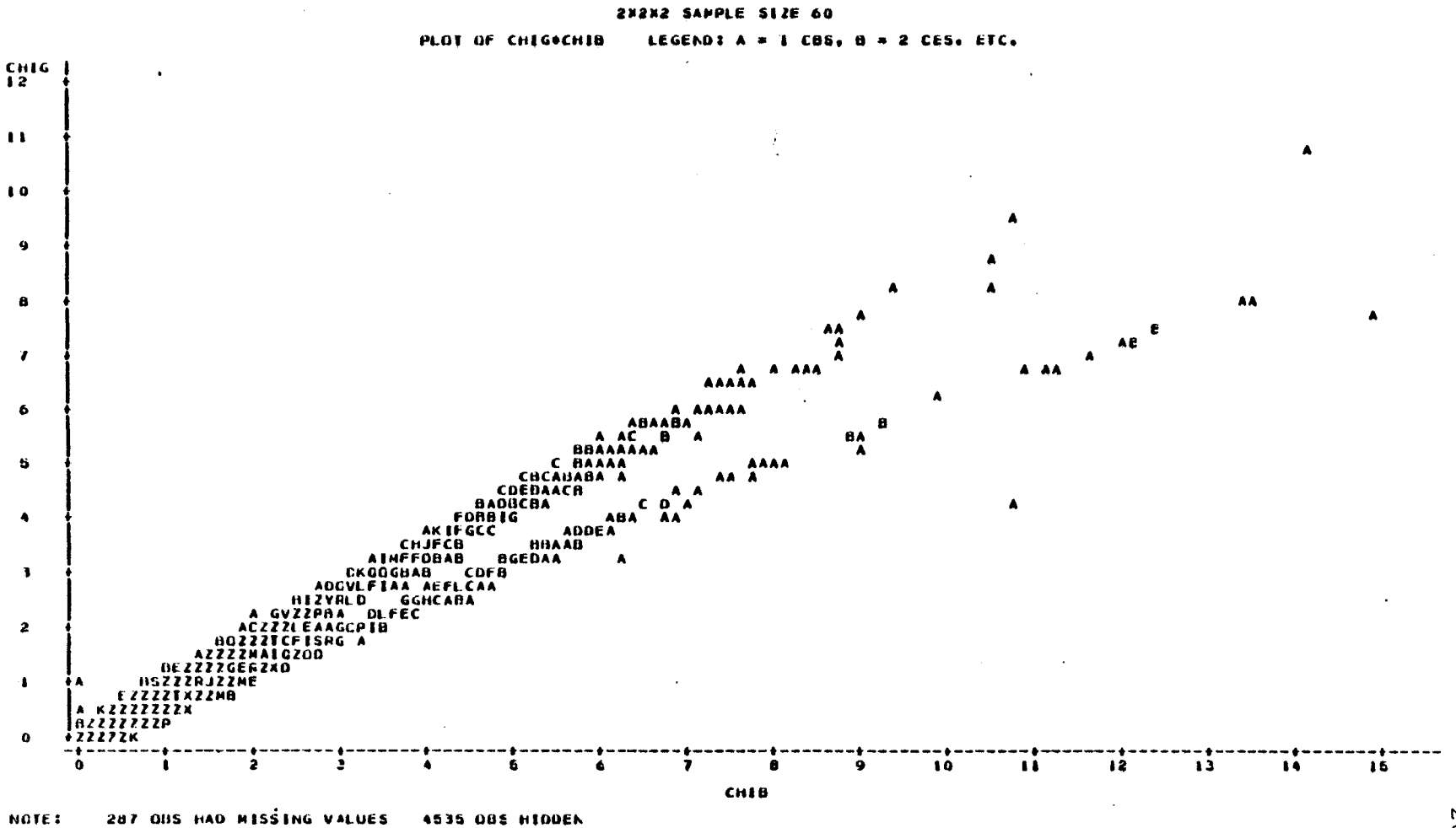

Figure 75. Scatter Diagram indicating two lines: Bartlett vs. Goodman 2

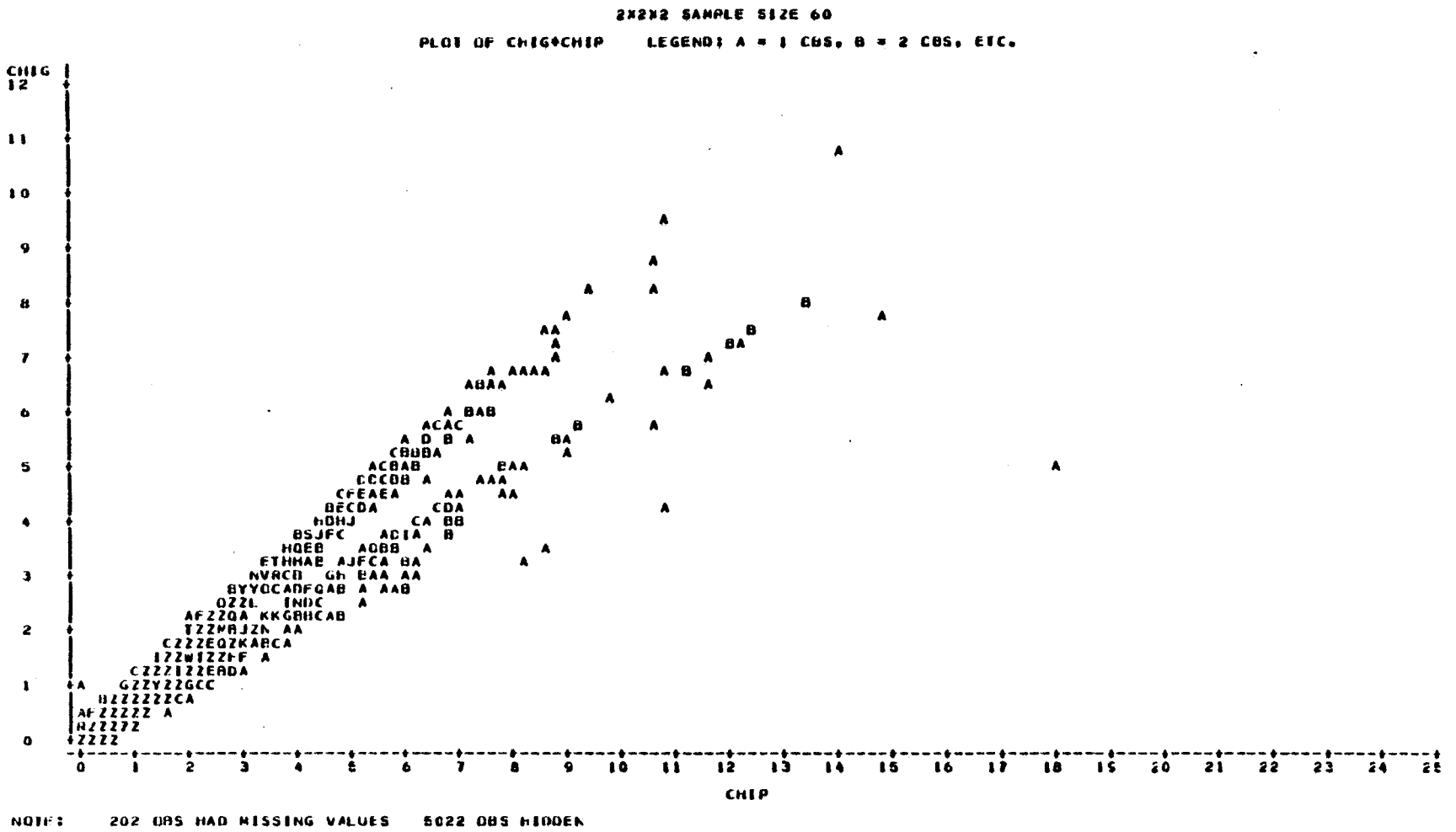

Figure 76. Scatter Diagram indicating two lines: IPF vs. Goodman 2

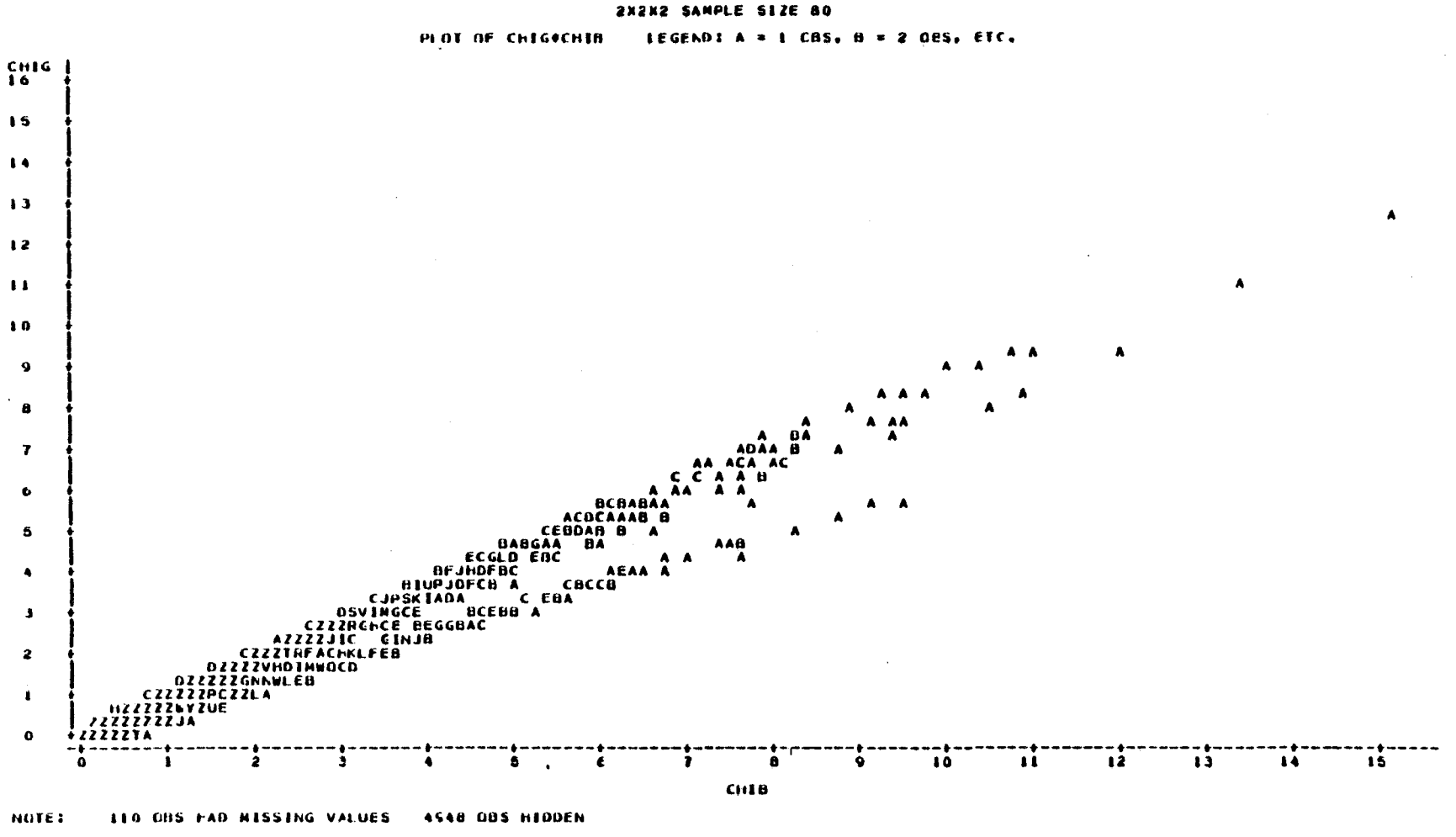

Figure 77. Scatter Diagram indicating two lines: Bartlett vs. Goodman 2

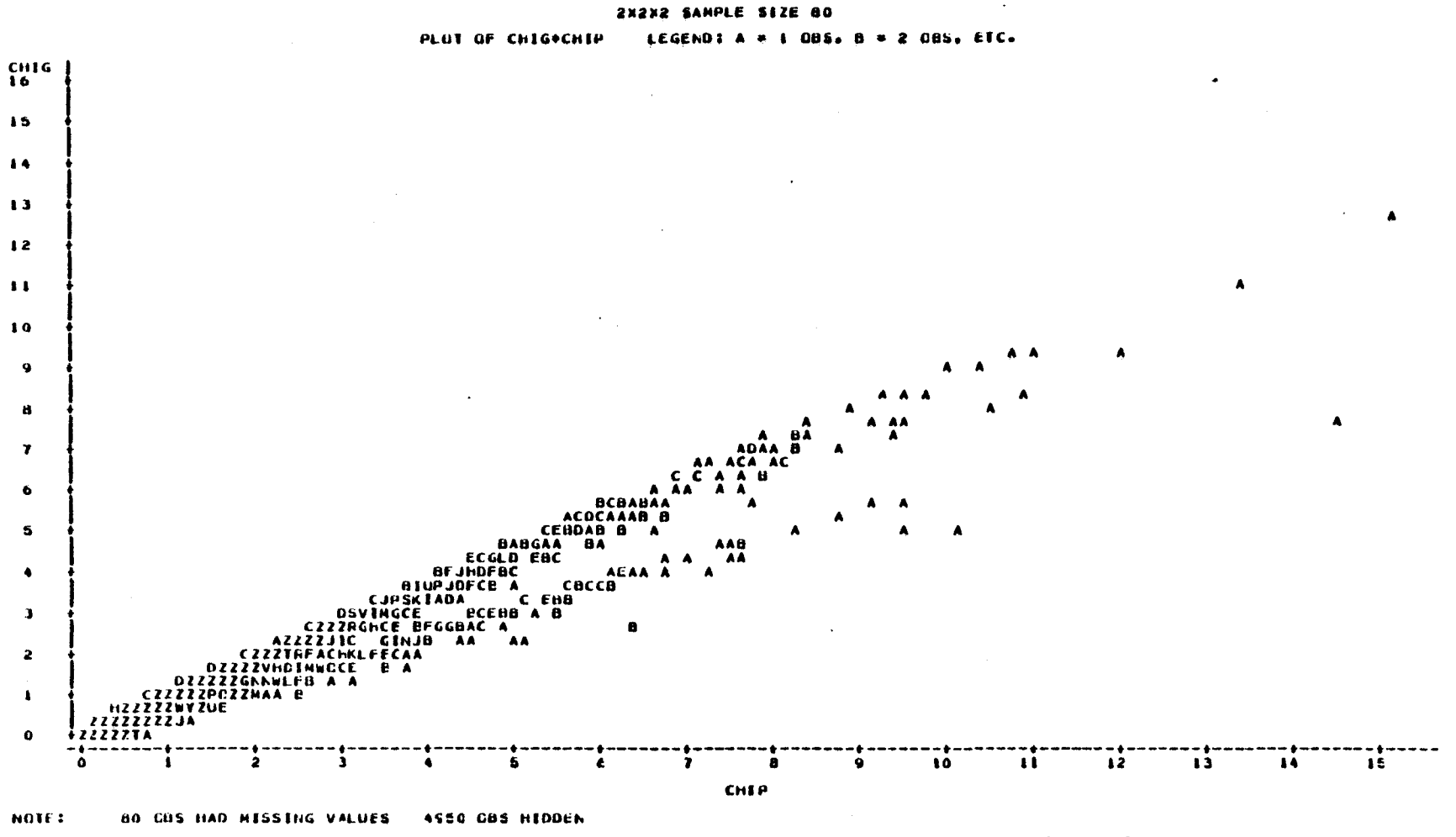

Figure 78. Scatter Diagram indicating two lines: IPF vs. Goodman 2

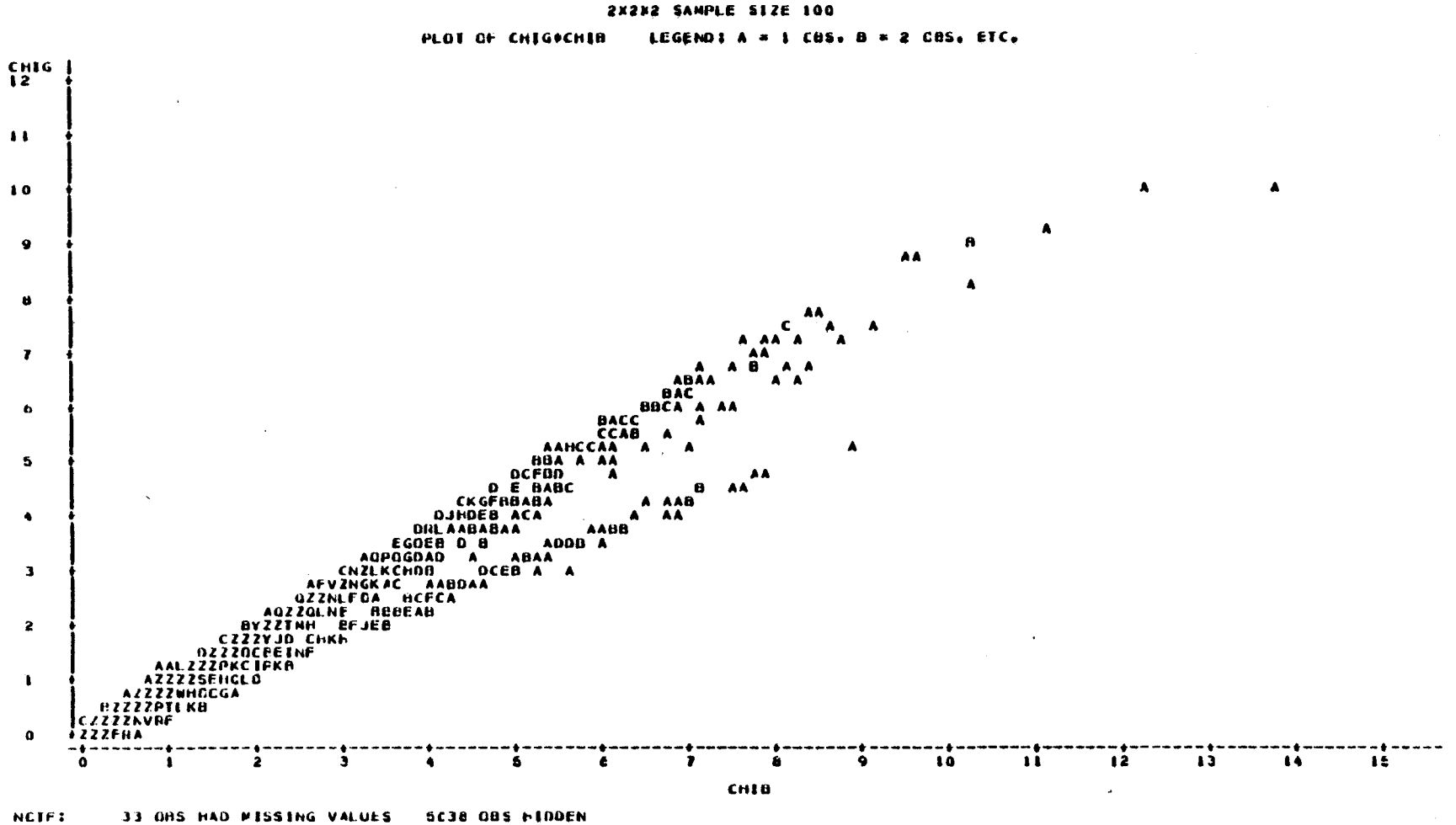

Figure 79. Scatter Diagram indicating two lines: Bartlett vs. Goodman 2

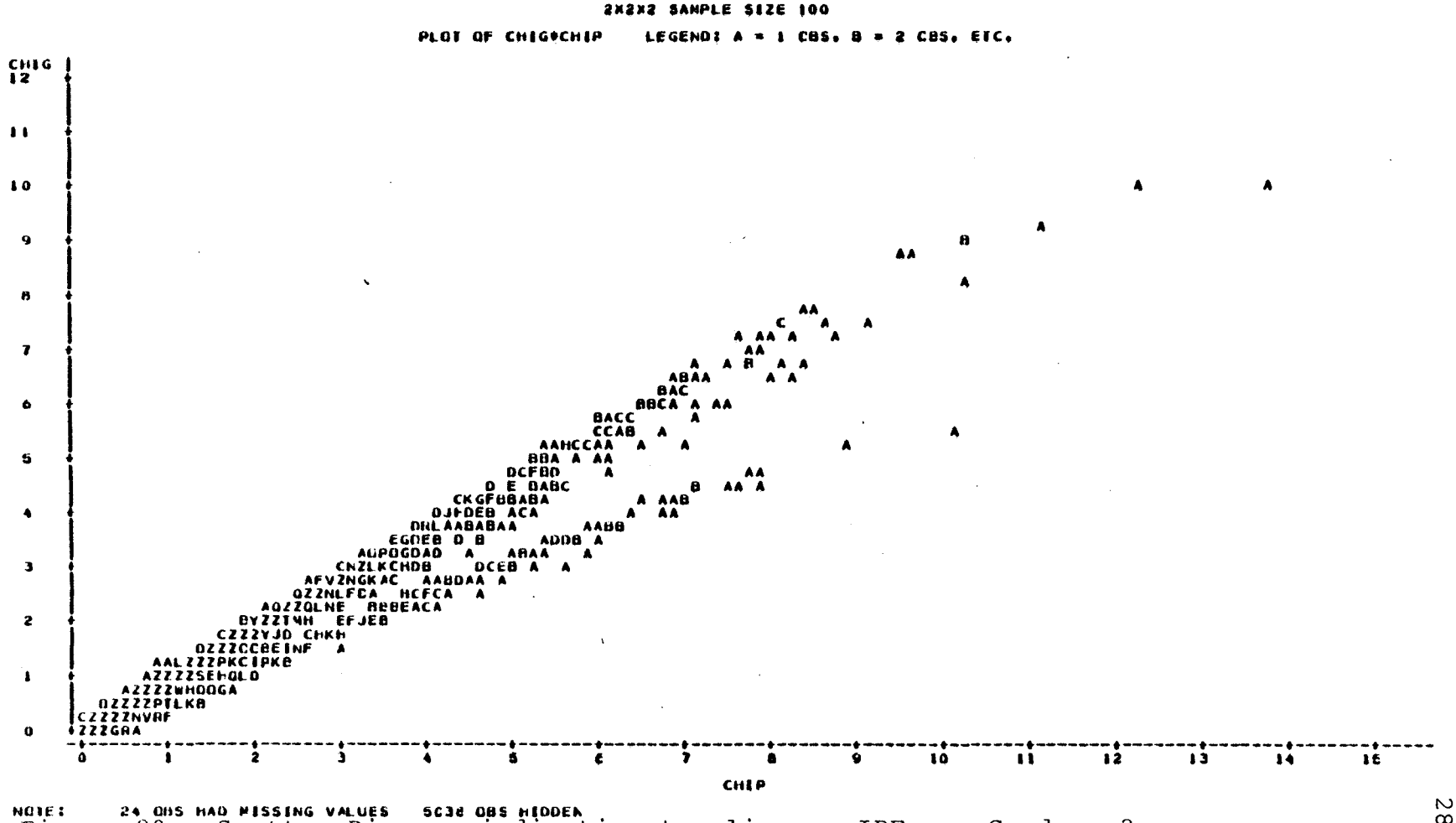

Figure 80. Scatter Diagram indicating two lines: IPF vs. Goodman 2

APPENDIX D

```
      DIMENSION X(3,3,12),GOO(4,12),B(3,12,12,2),XS(3,3,12),NGOO(3)
      DIMENSION D(3,3,12),M4(4,4,12),XA(3,3,12),IX(3,3,12),MK(3)
      DIMENSION AMS(12,12),J(4,4),GOOM(4),Y1(4),A(12,12),ESUM(3,3,12)
      DIMENSION IPROC(3),IPROL(3),IPROR(3),I1(3),T2(9),T3(27),IT3(27)
      DIMENSION AA(12,12),BB(12),CL(12,12),XX(12),ICHR(3,3,3)
      DOUBLE PRECISION X,GOO,S,SSUM,J,Q,AM,AMS,A,GOOM,Y1,Y,YY,CSGG,ANUM
      DOUBLE PRECISION DL,XA,C1,C2,C3,C4,CO,XJ,ATEMP,DET1,CSGP,CSGE,XDEN
      DOUBLE PRECISION AA,BB,CL,XX,DIGITS,CHI,Z,ZP
      CALL ERRSET(207,300,-1,2,0,209)
       IRN=1
      READ(5,3)IS,JS,LR,IDIMB,IDIME,NPROPB,NPROPE
      LR1=LR-1
      DO 2000 IDIM=IDIMB,IDIME
      READ(5,1) I1,J1,K1,CHI
      WRITE(6,4) I1,J1,K1
C  IDF=DEGREES OF FREEDOM
      IDF=(I1-1)*(J1-1)*(K1-1)
      WRITE(6,5) IDF,CHI
      IJK=I1*J1*K1
      DO 2000 NPROPS=NPROPB,NPROPE
      READ(5,2) (IPROR(I),I=1,I1)
      READ(5,2)(IPROC(J),J=1,J1)
      READ(5,2)(IPROL(K),K=1,K1)
      WRITE(6,7)(IPROR(I),I=1,I1),(IPROC(I),I=1,J1),(IPROL(I),I=1,K1)
      CALL RSTART(IS,JS)
      GREJ=0.
      PREJ=0.
      BREJ=0.
      SL=0.
      SC=0.
      SR=0.
      DO 410 K=1,K1
  410 SL=SL+IPROL(K)
      DO 411 J=1,J1
  411 SC=SC+IPROC(J)
      DO 412 I=1,I1
  412 SR=SR+IPROR(I)
      K2=K1-1
      DO 420 K=1,K2
      IF(K.NE.1) GO TO 421
      T1(K)=IPROL(1)/SL
      GO TO 420
  421 I4=K-1
      T1(K)=T1(I4)+IPROL(K)/SL
  420 CONTINUE
      T1(K1)=1.
      IT=0
      J2=J1-1
      DO 450 K=1,K1
      IF(IT.NE.0) GO TO 451
      DO 430 J=1,J2
      IT=IT+1
      IF(J.NE.1) GO TO 431
      T2(IT)=IPROC(1)*T1(1)/SL
      GO TO 430
  431 J4=IT-1
      T2(IT)=T2(J4)+IPROC(J)*T1(1)/SL
  430 CONTINUE
      IT=IT+1
      T2(IT)=T1(1)
      GO TO 450
  451 T=T1(K)-T1(K-1)
```

```
      DO 440 J=1,J2
      IT=IT+1
440   T2(IT)=T2(IT-1)+IPROC(J)*T/SC
      IT=IT+1
      T2(IT)=T1(K)
450   CONTINUE
      IR=0
      I2=I1-1
      DO 480 K=1,IT
      IF(IR.NE.0) GO TO 481
      DO 460 I=1,I2
      IR=IR+1
      IF(I.NE.1) GO TO 461
      T3(IR)=IPROR(1)*T2(1)/SC
      GO TO 460
461   I4=IR-1
      T3(IR)=T3(I4)+IPROR(I)*T2(1)/SC
460   CONTINUE
      IR=IR+1
      T3(IR)=T2(1)
      GO TO 480
481   T=T2(K)-T2(K-1)
      DO 470 I=1,I2
      IR=IR+1
470   T3(IR)=T3(IR-1)+IPROR(I)*T/SC
      IR=IR+1
      T3(IR)=T2(K)
480   CONTINUE
      NT=1625
      NG=NT
      NP=NG
      NB=NG
      DO 1500 NTABLE=1,NT
      DO 485 I=1,IJK
485   IT3(I)=0
      DO 490 L1=1,LR
      Z=UNI(0)
      IF(NTABLE.NE.NT) GO TO 486
      IF(L1.NE.LR1) GO TO 487
      IS=Z*10**6
      GO TO 486
487   IF(L1.NE.LR) GO TO 486
      JS=Z*10**6
486   DO 495 I=1,IJK
      IF(Z.LT.T3(I)) GO TO 490
495   CONTINUE
490   IT3(I)=IT3(I)+1
       IC=0
       IEMP=0
       IZE=0
       ITZP=0
       DO 900 K=1,K1
       DO 900 J=1,J1
       DO 900 I=1,I1
       IC=IC+1
       X(I,J,K)=IT3(IC)
       IX(I,J,K)=X(I,J,K)
       IF(IX(I,J,K).EQ.0) IEMP=IEMP+1
900    X3(I,J,K)=X(I,J,K)
       DO 911 I=1,I1
       DO 911 J=1,J1
       DO 911 K=1,K1
```

```
 511    IERR(I,J,K)=0
        IF(IEMP.LE.1) GO TO 509
        CALL DEGFM(I1,J1,K1,IX,IERR,IZE,ITZP)
 509    IFRMP=0
        IFRMG=0
        NREJ=0
C ITERATIVE PROPORTIONAL FITTING METHOD
C FORM X+JK'S OVER J,K
       DO 510 J=1,J1
       DO 510 K=1,K1
        GDD(J,K)=0
       DO 510 I=1,I1
 510    GDD(J,K)=GDD(J,K)+X8(I,J,K)
C FORM X-I+K'S OVER I,K
       DO 520 I=1,I1
       DO 520 K=1,K1
       AMS(I,K)=0
       DO 520 J=1,J1
 520   AMS(I,K)=AMS(I,K)+X8(I,J,K)
C FROM X-IJ+'S OVER I,J
       DO 530 I=1,I1
       DO 530 J=1,J1
       Q(I,J)=0
       DO 530 K=1,K1
 530   Q(I,J)=Q(I,J)+X8(I,J,K)
C SET EACH INITIAL APPROXIMATION TO 1
       DO 540 I=1,I1
       DO 540 J=1,J1
       DO 540 K=1,K1
 540   BSUM(I,J,K)=1
C IC=THE NUMBER OF COMPLETE 3-CYCLES NEED TO GET THE DIFFERENCE
C BETWEEN GIVEN CELL VALUES AND CELL EXPECTANCIES TO BE LESS THAN
C A PRE-ASSIGNED DL.  IN THIS CASE, DL=.0000001
       ICC=0
C SET UP WORKING ARRAY AM(I,J,K)
       DO 550 I=1,I1
       DO 550 J=1,J1
       DO 550 K=1,K1
 550   AM(I,J,K)=BSUM(I,J,K)
C FORM X-IJ+'S ESTIMATES OVER I,J
 555   DO 560 I=1,I1
       DO 560 J=1,J1
       A(I,J)=0
       DO 560 K=1,K1
 560   A(I,J)=A(I,J)+AM(I,J,K)
C FORM 1ST CYCLE CELL EXPECTANCIES
       DO 570 I=1,I1
       DO 570 J=1,J1
       DO 570 K=1,K1
       IF (A(I,J).NE.0.) GO TO 571
       AM(I,J,K)=0.
       GO TO 570
 571   AM(I,J,K)=AM(I,J,K)*Q(I,J)/A(I,J)
 570   CONTINUE
C FORM X-I+K'S ESTIMATES OVER I,K
       DO 580 I=1,I1
       DO 580 K=1,K1
       A(I,K)=0
       DO 580 J=1,J1
 580   A(I,K)=A(I,K)+AM(I,J,K)
C FROM 2ND CYCLE CELL EXPECTANCIES USING THE 1ST CYCLE CELL
C ESTIMATES
```

```
      DO 590 I=1,I1
      DO 590 J=1,J1
      DO 590 K=1,K1
      IF(A(I,K).NE.0.) GO TO 591
      AM(I,J,K)=0.
      GO TO 590
 591  AM(I,J,K)=AM(I,J,K)*AMS(I,K)/A(I,K)
      CALL DVCHK(IDIV)
      CALL OVERFL(IFLOW)
      IF(IDIV .NE.2 .OR. IFLOW .NE.2) GO TO 626
 590  CONTINUE
C FROM X+JK'S ESTIMATES OVER J,K
      DO 600 J=1,J1
      DO 600 K=1,K1
      A(J,K)=0
      DO 600 I=1,I1
 600   A(J,K)=A(J,K)+AM(I,J,K)
C FORM 3RD CYCLE CELL EXPECTANCIES USING THE 2ND CYCLE CELL
C ESTIMATES
      DO 610 I=1,I1
      DO 610 J=1,J1
      DO 610 K=1,K1
      IF(A(J,K).NE.0.) GO TO 611
      AM(I,J,K)=0.
      GO TO 610
 611   AM(I,J,K)=AM(I,J,K)*GOO(J,K)/A(J,K)
 610   CONTINUE
C FIND DIFFERENCES BETWEEN GIVEN AND EXPECTED CELL VALUES. IF
C EACH IS NOT LESS THAN DT, THEN GO TO 633 TO START REPEATING
C THE PROCEDURE FOR THE NEXT CYCLE.  IF EACH IS LESS THAN DT,
C COMPUTE CHI-SQUARE USING PEARSON STATISTIC
      DO 620 I=1,I1
      DO 620 J=1,J1
      DO 620 K=1,K1
      DL=BSUM(I,J,K)-AM(I,J,K)
      IF(DABS(DL) .GE. .000001) GO TO 633
 620   CONTINUE
      IF(IEMP.LE.1) GO TO 621
      DO 625 I=1,I1
      DO 625 J=1,J1
      DO 625 K=1,K1
      IF(IX(I,J,K).EQ.0.AND.IERR(I,J,K).EQ.0.AND.DABS(AM(I,J,K)).LT..000
     5001) GO TO 624
      GO TO 625
 624  IZE=IZE+1
      IERR(I,J,K)=1
 625  CONTINUE
 621  IADF=IDF-IZE+ITZP
      IF(IADF.GT.0) GO TO 622
 626  IFRMP=100
      NIP=NIP-1
      GO TO 700
C COMPUTE PEARSON STATISTIC
 622  CSQP=0.
      DO 630 I=1,I1
      DO 630 J=1,J1
      DO 630 K=1,K1
      IF(IERR(I,J,K).EQ.1) GO TO 630
      ZP=(X8(I,J,K)-AM(I,J,K))**2/AM(I,J,K)
      CALL DVCHK(IDIV)
      CALL OVERFL(IFLOW)
      IF(IDIV .NE. 2 .OR. IFLOW .NE. 2) GO TO 626
```

```
      CSQP=CSQP + ZP
  650 CONTINUE
      GO TO (651,652,653,654,655,656,657,658),IADF
  651 IF(3.84.GT.CSQP) GO TO 700
      GO TO 632
  652 IF(5.99.GT.CSQP) GO TO 700
      GO TO 632
  653 IF(7.81.GT.CSQP) GO TO 700
      GO TO 632
  654 IF(9.49.GT.CSQP) GO TO 700
      GO TO 632
  655 IF(11.07.GT.CSQP) GO TO 700
      GO TO 632
  656 IF(12.59.GT.CSQP) GO TO 700
      GO TO 632
  657 IF(14.07.GT.CSQP) GO TO 700
      GO TO 632
  658 IF(15.51.GT.CSQP) GO TO 700
  632 NREJ=NREJ+1
      PREJ=PREJ+1
       GO TO 700
C SET 3RD CYCLE ESTIMATES AS NEW ESTIMATES SO THAT THE NEXT
C 3-CYCLE PROCEDURE CAN BE STARTED
  633  DO 640 I=1,I1
       DO 640 J=1,J1
      DO 640 K=1,K1
  640 BSUM(I,J,K)=AM(I,J,K)
C KICK UP THEN NUMBER OF 3-CYCLE PROCEDURES COMPLETED
      ICC=ICC+1
C GO BACK TO BEGIN THE NEXT CYCLE
      GO TO 555
C BARTLETT METHOD FOLLOWS
  700 NCHB=0
  701  M=0
       IFRMD=0
       DO 710 I=1,I1
      DO 710 J=1,J1
      DO 710 K=1,K1
      D(I,J,K)=0.
      XB(I,J,K)=X(I,J,K)
  710 XA(I,J,K)=0.
      IS=I1-1
      JS=J1-1
      KS=K1-1
  715 M=M+1
      DO 740 I=1,IS
      DO 740 J=1,JS
      IF(M.NE.1) GO TO 716
      DO 717 K=1,K1
      IF(XB(I,J1,K)*XB(I1,J,K).EQ.0..OR.XB(I1,J1,K)*XB(I,J,K).EQ.0.) GO
     5TO 721
      NADJ(K)=0
      GO TO 717
  721 NADJ(K)=1
  717 CONTINUE
      NADJC=0
      DO 718 K=1,K1
      IF(NADJ(K).EQ.1) NADJC=NADJC+1
  718 CONTINUE
      IF(NADJC.EQ.0) GO TO 716
      IF(NADJC.LT.K1) GO TO 719
      GO TO 860
```

```
719   CALL ADJUST(I,J,K1,NADJ,D,NADJC,II,JJ,X)
      GO TO 731
715   DO 720 K=1,K1
      IF(XB(II,JJ,K)*XB(I,J,K).NE.0.) GO TO 727
      GO TO 860
727   IF(DABS(XB(II,JJ,K)).GT.50..AND.DABS(XB(I,J,K)).GT.50..AND.DABS(XB
     B(I,JJ,K)).GT.50..AND.DABS(XB(II,J,K)).GT.50.) GO TO 860
      BSUM(I,J,K)=(XB(I,JJ,K)*XB(II,J,K))/(XB(II,JJ,K)*XB(I,J,K))
       C1=XB(I,J,K)*XB(I,JJ,K)*XB(II,J,K)
       C2=XB(II,JJ,K)*XB(I,JJ,K)*XB(II,J,K)
       C3=XB(II,JJ,K)*XB(I,J,K)*XB(II,J,K)
       C4=XB(II,JJ,K)*XB(I,J,K)*XB(I,JJ,K)
       CD=XB(II,JJ,K)*XB(I,J,K)*XB(I,JJ,K)*XB(II,J,K)
       IF(C1+C2+C3+C4.NE.0.) GO TO 725
       GO TO 860
725   AM(I,J,K)=CD/(C1+C2+C3+C4)
       CALL OVCHK(IDIV)
       CALL OVERFL(IFLOW)
       IF(IFLOW.NE.2) GO TO 860
       IF(IDIV .NE. 2 .OR. IFLOW .NE. 2) GO TO 860
720   CONTINUE
      Y=0
      YY=0
      DO 730 K=1,K1
      Y=Y+AM(I,J,K)
730    YY=YY+BSUM(I,J,K)*AM(I,J,K)
      Q(I,J)=Y/YY
      CALL OVCHK(IDIV)
      CALL OVERFL(IFLOW)
      IF(IDIV .NE. 2 .OR. IFLOW .NE. 2) GO TO 860
      DO 735 K=1,K1
735   D(I,J,K)=AM(I,J,K)*(1-Q(I,J)*BSUM(I,J,K))
731   DO 740 K=1,K1
      XA(I,J,K)=XA(I,J,K)+D(I,J,K)
      XB(II,JJ,K)=XB(II,JJ,K)-D(I,J,K)
      XB(I,J,K)=XB(I,J,K)-D(I,J,K)
      XB(I,JJ,K)=XB(I,JJ,K)+D(I,J,K)
740    XB(II,J,K)=XB(II,J,K)+D(I,J,K)
      DO 750 I=1,IS
      DO 750 J=1,JS
      DO 750 K=1,K1
      IF(DABS(D(I,J,K)).GE..000001) GO TO 715
750    CONTINUE
      D(II,JJ,K1)=0
      DO 770 II=1,IS
      DO 770 JJ=1,JS
      DO 770 KK=1,KS
770   D(II,JJ,K1)=D(II,JJ,K1)+XA(II,JJ,KK)
      DO 780 I=1,IS
      D(I,JJ,K1)=0
      DO 775 JJ=1,JS
      DO 775 KK=1,KS
775   D(I,JJ,K1)=D(I,JJ,K1)+XA(I,JJ,KK)
780   CONTINUE
      DO 790 J=1,JS
      D(II,J,K1)=0
      DO 785 II=1,IS
      DO 785 KK=1,KS
785    D(II,J,K1)=D(II,J,K1)+XA(II,J,KK)
790    CONTINUE
      DO 800 K=1,KS
      D(II,JJ,K)=0
```

```
      DO 795 II=1,IS
      DO 795 JJ=1,JS
795    D(II,JJ,K)=D(II,JJ,K)+XA(II,JJ,K)
800   CONTINUE
      DO 810 I=1,IS
      DO 810 J=1,JS
      D(I,J,K1)=0
      DO 805 KK=1,KS
805   D(I,J,K1)=D(I,J,K1)+XA(I,J,KK)
810   CONTINUE
      DO 820 I=1,IS
      DO 820 K=1,KS
      D(I,J1,K)=0
      DO 815 JJ=1,JS
815   D(I,J1,K)=D(I,J1,K)+XA(I,JJ,K)
820   CONTINUE
      DO 830 J=1,JS
      DO 830 K=1,KS
      D(I1,J,K)=0
      DO 825 II=1,IS
825   D(II,J,K)=D(II,J,K)+XA(II,J,K)
830   CONTINUE
      DO 840 I=1,IS
      DO 840 J=1,JS
      DO 840 K=1,KS
840   D(I,J,K)=XA(I,J,K)
      CSQB=0.
      DO 850 I=1,I1
      DO 850 J=1,J1
      DO 850 K=1,K1
      IF(I.NE.I1.AND.J.NE.J1.AND.K.NE.K1) GO TO 855
      IF (I.NE.I1.AND.J.EQ.J1.AND.K.EQ.K1) GO TO 855
      IF (I.EQ.I1.AND.J.NE.J1.AND.K.EQ.K1) GO TO 855
      IF (I.EQ.I1.AND.J.EQ.J1.AND.K.NE.K1) GO TO 855
      N=1
      GO TO 856
 855  N=-1
 856  Z=D(I,J,K)**2/(X(I,J,K)+N*D(I,J,K))
      CALL DVCHK(IDIV)
      CALL OVERFL(IFLG*)
      IF(IDIV .NE. 2 .OR. IFLG* .NE.2) GO TO 860
     IF (Z .LT. 0.) GO TO 853
 850  CSQB=CSQB+Z
      IF(CSQB.GE.0.) GO TO 851
853   IF(NCHB .LT. IS**3) GO TO 865
      NB=NB-1
      GO TO 870
851   IF(CHI.GT.CSQB) GO TO 870
      NREJ=NREJ+1
     BREJ=BREJ+1.
     GO TO 870
860   IF(NCHB.LT.IS**3) GO TO 865
     IFRM=10
     NB=NB-1
     GO TO 870
865   CALL BARTCH(I1,J1,K1,X,NCHB)
     GO TO 701
870   DO 875 I=1,I1
     DO 875 J=1,J1
     DO 875 K=1,K1
875   X(I,J,K)=IX(I,J,K)
C  GOODMAN METHOD FOLLOWS
```

```
 29     NCHG=0
        DO 41 K=1,K1
        DO 41 J=1,J1
        DO 41 I=1,I1
        X(I,J,K)=X(I,J,K)+.5
 41     CONTINUE
        DO 45 K=1,K1
        DO 45 J=1,J1
        DO 45 I=1,I1
        AM(I,J,K)=ALOG(X(I,J,K)*X(I1,J1,K)/(X(I,J1,K)*X(I1,J,K)))
        CALL DVCHK(IDIV)
        CALL OVERFL(IFLOW)
        IF(IDIV .NE. 2 .OR. IFLOW .NE. 2) GO TO 178
 45     CONTINUE
        KI1=I5*J5
C FORM: G-DOT-DOT-K'S
        DO 20 K=1,K1
        M=0
        DO 20 I=1,I5
        DO 20 J=1,J5
        M=M+1
 20     GDD(M,K)=AM(I,J,K)
C FORM B MATRICES FOR EACH I AND K
        DO 30 K=1,K1
        DO 30 I=1,I1
        SUM=0.
        DO 40 J=1,J1
 40     SUM=X(I,J,K)+SUM
        DO 30 J=1,J5
        DO 30 JJ=1,J5
        IF(J.EQ.JJ) GO TO 35
        B(I,K,J,JJ)=-X(I,J,K)*X(I,JJ,K)/SUM
        CALL DVCHK(IDIV)
        CALL OVERFL(IFLOW)
        IF(IDIV .NE. 2 .OR. IFLOW .NE. 2) GO TO 178
        GO TO 30
 35     B(I,K,J,JJ)=X(I,J,K)*(1.-X(I,J,K)/SUM)
        CALL DVCHK(IDIV)
        CALL OVERFL(IFLOW)
        IF(IDIV .NE. 2 .OR. IFLOW .NE. 2) GO TO 178
 30     CONTINUE
C FORM SUMS OF 3 MATRICES FOR EACH K
        DO 50 K=1,K1
        DO 50 J=1,J5
        DO 50 JJ=1,J5
        BSUM(J,JJ,K)=0.
        DO 50 I=1,I1
 50     BSUM(J,JJ,K)=B(I,K,J,JJ)+BSUM(J,JJ,K)
C GET D-K'S=INVERSES OF EACH SUM JUST COMPUTED
        GO TO (55,65),J5
 55     DO 60 K=1,K1
        IF(BSUM(1,1,K).NE.0.) GO TO 60
        IFRMG=1
        NG=NG-1
        GO TO 271
 60      D(1,1,K)=1./BSUM(1,1,K)
        CALL DVCHK(IDIV)
        CALL OVERFL(IFLOW)
        IF(IDIV .NE. 2 .OR. IFLOW .NE. 2) GO TO 178
        GO TO 75
 65     DO 70 K=1,K1
        IFLAG=0
```

```
      DO 67 J=1,JS
      DO 67 I=1,JS
 67   AA(I,J)=BSUM(I,J,K)
      CALL DECOMP(JS,AA,UL,IFLAG)
      IF(IFLAG.EQ.0) GO TO 66
      IFRMG=1
      NG=NG-1
      GO TO 271
 66   DO 70 J=1,JS
      DO 68 I=1,JS
      IF(I.EQ.J) GO TO 69
      BB(I)=0.
      GO TO 68
 69   BB(I)=1.
 68   CONTINUE
      CALL SOLVE(JS,UL,BB,XX,IFLAG)
      IF(IFLAG.EQ.1) GO TO 78
      CALL IMPROV(JS,AA,UL,BB,XX,DIGITS,IFLAG)
      IF(IFLAG.EQ.0) GO TO 71
 78   IFRMG=1
      NG=NG-1
      GO TO 271
 71   DO 70 I=1,JS
 70   D(I,J,K)=XX(I)
C GET M-K'S WHICH ARE I1-1 SQUARE MATRICES WHERE EACH ENTRY IS
C A J1-1 SQUARE MATRIX, B
 75   DO 80 K=1,K1
      DO 80 I=1,IS
      DO 80 IH=1,IS
      IF(I.NE.IH) GO TO 125
      DO 90 J=1,JS
      DO 90 JJ=1,JS
      A(J,JJ)=0
      DO 90 KK=1,JS
      A(J,JJ)=A(J,JJ)+B(I,K,J,KK)*D(KK,JJ,K)
 90   CONTINUE
      DO 100 J=1,JS
      DO 100 JJ=1,JS
      Q(J,JJ)=0
      DO 100 KK=1,JS
      Q(J,JJ)=Q(J,JJ)+A(J,KK)*B(I,K,KK,JJ)
 100  CONTINUE
          DO  110  J=1,JS
          DO  110  JJ=1,JS
 110  Q(J,JJ)=B(I,K,J,JJ)-Q(J,JJ)
      L1=(I-1)*JS+1
      L2=(IH-1)*JS+1
      L3=I*JS
      L4=IH*JS
      J=0
      DO 120 L=L1,L3
      J=J+1
      JJ=0
      DO 120 LL=L2,L4
      JJ=JJ+1
 120  AM(L,LL,K)=Q(J,JJ)
      GO TO 80
 125  DO 130 J=1,JS
      DO 130 JJ=1,JS
      A(J,JJ)=0
      DO 130 KK=1,JS
 130  A(J,JJ)=A(J,JJ)+B(I,K,J,KK)*D(KK,JJ,K)
```

```
      DO 140 J=1,JS
      DO 140 JJ=1,JS
      Q(J,JJ)=0
      DO 140 KK=1,JS
      Q(J,JJ)=Q(J,JJ)+A(J,KK)*B(IH,K,KK,JJ)
      CALL OVERFL(IFLG*)
      IF(IFLG* .NE.2) GO TO 178
  140 CONTINUE
      DO 145 M3=1,JS
      DO 145 M4=1,JS
  145 Q(M3,M4)=-Q(M3,M4)
      L1=(I-1)*JS+1
      L2=(IH-1)*JS+1
      L3=I*JS
      L4=IH*JS
      J=0
      DO 150 L=L1,L3
      J=J+1
      JJ=0
      DO 150 LL=L2,L4
      JJ=JJ+1
  150 AM(L,LL,K)=Q(J,JJ)
   80 CONTINUE
C ADD THE M-K MATRICES
      DO 160 I=1,KII
      DO 160 J=1,KII
      AMS(I,J)=0
      DO 160 K=1,KI
  160 AMS(I,J)=AMS(I,J)+AM(I,J,K)
C FORM Q WHICH IS THE INVERSE OF THE SUMS JUST FOUND
      IF(KII.NE.1) GO TO 165
  163 IF(AMS(1,1).NE.0.) GO TO 164
      IFRMG=1
      NG=NG-1
      GO TO 271
  164 Q(1,1)=1./AMS(1,1)
      CALL DVCHK(IDIV)
      CALL OVERFL(IFLG*)
      IF(IDIV .NE. 2 .OR. IFLG* .NE. 2) GO TO 178
      GO TO 185
  165 IFLAG=0
      CALL DECOMP(KII,AMS,UL,IFLAG)
      IF(IFLAG.EQ.0) GO TO 166
      IFRMG=1
      NG=NG-1
      GO TO 271
  166 DO 170 J=1,KII
      DO 175 I=1,KII
      IF(I.EQ.J) GO TO 168
      BB(I)=0.
      GO TO 175
  168 BB(I)=1.
  175 CONTINUE
      CALL SOLVE(KII,UL,BB,XX,IFLAG)
      IF(IFLAG.EQ.1) GO TO 178
      CALL IMPROV(KII,AMS,UL,BB,XX,DIGITS,IFLAG)
      IF(IFLAG.EQ.0) GO TO 176
  178 IFRMG=1
      NG=NG-1
      GO TO 271
  176 DO 170 I=1,KII
  170 Q(I,J)=XX(I)
```

```
  195   DO 190 I=1,KII
C FORM G-HAT'S FOR EACH K
  190   GDDH(I)=0
        DO 210 K=1,K1
        DO 200 I=1,KII
        A(I,K)=0
        DO 200 J=1,KII
  200   A(I,K)=A(I,K)+AM(I,J,K)*GDD(J,K)
C FORM G-HAT'*Q*G-HAT FOR EACH K
        DO 210 I=1,KII
  210   GDDH(I)=GDDH(I)+A(I,K)
        DO 220 K=1,KII
        Y1(K)=0
        DO 220 J=1,KII
  220   Y1(K)=Y1(K)+GDDH(J)*Q(J,K)
C ADD THE ABOVE OVER ALL K
        Y=0
        DO 230 I=1,KII
  230   Y=Y+Y1(I)*GDDH(I)
        DO 240 I=1,K1
        DO 240 J=1,KII
  240   A(I,J)=0
C FORM G-DOT-DOT-K'*M-K*G-DOT-DOT-K FOR EACH K
        YY=0
        DO 270 K=1,K1
        Y2=0
        DO 250 J=1,KII
        DO 250 I=1,KII
  250   A(K,J)=GDD(I,K)*AM(I,J,K)+A(K,J)
C ADD THE ABOVE PRODUCTS OVER ALL K
        DO 260 J=1,KII
  260   Y2=Y2+A(K,J)*GDD(J,K)
  270   YY=YY+Y2
C FIND CHI-SQUARE
        CSQG=YY-Y
        IF(CHI.GT.CSQG) GO TO 271
        NREJ=NREJ+1
        GREJ=GREJ+1
  271   IFMT=IFRMP+IFRMB+IFRMG+3000
        IF(IFMT.EQ.3000) GO TO 1100
 1102 CALL FORMT(IFMT,NTABLE,IX,CSQP,CSQB,CSQG,I1,J1,K1,IFRMP,IFRMB,IFRM
     7G,NPROPS,LR,NREJ,IADF,IDIM)
        GO TO 1500
 1100 IF(NREJ.EQ.3.OR.NREJ.EQ.0) GO TO 1400
        CALL FORMT(IFMT,NTABLE,IX,CSQP,CSQB,CSQG,I1,J1,K1,IFRMP,IFRMB,IFRM
     7G,NPROPS,LR,NREJ,IADF,IDIM)
          GO TO 1500
 1400   IF(CSQP.GE.0..AND.CSQB.GE.0..AND.CSQG.GT.-0.01) GO TO 1490
        IFMT=3000
        GO TO 1102
 1490 WRITE(8,9)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSQP,CS
     5QB,CSQG,(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
 1500 CONTINUE
         IPFREJ=PFREJ
         IBREJ=BREJ
         IGREJ=GREJ
         IF(NG.EQ.0.OR.NIP.EQ.0.OR.NB.EQ.0) GO TO 1501
         GO TO 1503
 1501   WRITE(6,19) IPFREJ,NIP,IBREJ,NB,IGREJ,NG
        WRITE(6,13)IS,JS
        GO TO 2000
 1503   REJG=GREJ/NG
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```
      REJPF=IPFREJ/NIP
      REJB=IBREJ/NB
      WRITE(6,8)REJPF,IPFREJ,NIP,REJB,IBREJ,NB,REJG,IGREJ,NG
      WRITE(6,13)IS,JS
2000  CONTINUE
1000  STOP
1     FORMAT(3I2,F5.2)
2     FORMAT(3I1)
3     FORMAT(2I6,I3,4I1)
4     FORMAT(//,20X,'DIM=',I2,2X,I2,2X,I2)
5     FORMAT(20X,'DF=',I2,5X,'CV=',F5.2,/)
7     FORMAT(20X,9I3)
8     FORMAT(2X,'IP:',2X,F6.4,2I5,3X,'B:',2X,F6.4,2I5,3X,'G:',2X,F6.4,2
     3I5,//)
9     FORMAT(3I2,5I3,I4,3(F7.2),2712)
13    FORMAT(5X,'IS=',I7,5X,'JS=',I7)
19    FORMAT(5X,'IP: ',2(I4,1X),'B: ',2(I4,1X),'G: ',2(I4,1X))
      END
      SUBROUTINE DEGFM(I1,J1,K1,IX,IERR,IZE,ITZP)
      DIMENSION IX(3,3,3),IERR(3,3,3),IJ(3,3),IK(3,3),JK(3,3)
      IJZP=0
      DO 20 I=1,I1
      DO 20 J=1,J1
      IJ(I,J)=0
      DO 25 K=1,K1
25    IJ(I,J)=IX(I,J,K)+IJ(I,J)
      IF(IJ(I,J).NE.0) GO TO 20
      DO 26 K=1,K1
26    IERR(I,J,K)=1
      IJZP=IJZP+1
20    CONTINUE
      IKZP=0
      DO 30 I=1,I1
      DO 30 K=1,K1
      IK(I,K)=0
      DO 35 J=1,J1
35    IK(I,K)=IX(I,J,K)+IK(I,K)
      IF(IK(I,K).NE.0) GO TO 30
      DO 36 J=1,J1
36    IERR(I,J,K)=1
      IKZP=IKZP+1
30    CONTINUE
      JKZP=0
      DO 40 J=1,J1
      DO 40 K=1,K1
      JK(J,K)=0
      DO 45 I=1,I1
45    JK(J,K)=IX(I,J,K)+JK(J,K)
      IF(JK(J,K).NE.0) GO TO 40
      DO 46 I=1,I1
46    IERR(I,J,K)=1
      JKZP=JKZP+1
40    CONTINUE
      IZP=0
      DO 50 I=1,I1
      IPP=0
      DO 55 J=1,J1
      DO 55 K=1,K1
55    IPP=IX(I,J,K)+IPP
      IF(IPP.EQ.0) IZP=IZP+1
50    CONTINUE
      IJZP=IJZP-IZP
```

```
      IKZP=IKZP-IZP
      JZP=0
      DO 60 J=1,J1
      JPP=0
      DO 65 I=1,I1
      DO 65 K=1,K1
65    JPP=JPP+IX(I,J,K)
      IF(JPP.EQ.0) JZP=JZP+1
60    CONTINUE
      IJZP=IJZP-JZP
      JKZP=JKZP-JZP
      KZP=0
      DO 70 K=1,K1
      KPP=0
      DO 75 I=1,I1
      DO 75 J=1,J1
75    KPP=KPP+IX(I,J,K)
      IF(KPP.EQ.0) KZP=KZP+1
70    CONTINUE
      IKZP=IKZP-KZP
      JKZP=JKZP-KZP
      IZE=0
      DO 80 I=1,I1
      DO 80 J=1,J1
      DO 80 K=1,K1
      IF(IERR(I,J,K).EQ.1) IZE=IZE+1
80    CONTINUE
      ITZP=IJZP+IKZP+JKZP+IZP+JZP+KZP
      RETURN
      END
      SUBROUTINE ADJUST(I,J,K1,NADJ,D,NADJC,I1,J1,X)
      DIMENSION NADJ(3),D(3,3,12),MK(3),X(3,3,12)
      DOUBLE PRECISION D,X
      K5=K1-1
      GO TO (10,20),K5
10    IF(NADJ(1).NE.1) GO TO 11
      D(I,J,1)=.75
      D(I,J,2)=-.75
      RETURN
11    D(I,J,1)=-.75
      D(I,J,2)=.75
      RETURN
20    IF(NADJC.NE.K5) GO TO 21
      DO 25 K=1,K1
      IF(NADJ(K).NE.1) GO TO 22
25    CONTINUE
22    DO 30 KK=1,K1
      IF(KK.EQ.K) GO TO 31
      D(I,J,K)=.75
      GO TO 30
31    D(I,J,K)=-1.5
30    CONTINUE
      RETURN
21    DO 35 I8=1,3
35    MK(I8)=0
      I9=0
      DO 40 K=1,K1
      IF(NADJ(K).EQ.1) GO TO 40
      I9=I9+1
      MK(I9)=K
40    CONTINUE
      K6=MK(1)
```

```
      K7=4K(2)
      ICK6=0
      ICK7=0
      DO 45 II=1,I1
      DO 45 JJ=1,J1
      IF(X(I,J,K6).GE.X(I,J,K7)) GO TO 46
      ICK7=ICK7+1
      GO TO 45
46    ICK6=ICK6+1
45    CONTINUE
      IF(ICK6.GE.ICK7) GO TO 51
      D(I,J,K7)=-.75
      DO 50 KK=1,K1
      IF(KK.EQ.K7.OR.NADJ(KK).NE.1) GO TO 50
      D(I,J,KK)=.75
50    CONTINUE
      RETURN
51    D(I,J,K6)=-.75
      DO 55 KK=1,K1
      IF(KK.EQ.K6.OR.NADJ(KK).NE.1) GO TO 55
      D(I,J,KK)=.75
55    CONTINUE
      RETURN
      END
      SUBROUTINE FORMT(IFMT,NTABLE,IX,CSQP,CSQB,CSQG,I1,J1,K1,IFRMP,IFRM
     8B,IFRMG,NPROPS,LR,NREJ,IADF,IDIM)
      DIMENSION IX(3,3,12)
      DOUBLE PRECISION CSQP,CSQB,CSQG
      IRN=2
      IF(IFMT.EQ.3000) GO TO 3000
      IF(IFMT.EQ.3001) GO TO 3001
      IF(IFMT.EQ.3010) GO TO 3010
      IF(IFMT.EQ.3100) GO TO 3100
      IF(IFMT.EQ.3011) GO TO 3011
      IF(IFMT.EQ.3101) GO TO 3101
      IF(IFMT.EQ.3110) GO TO 3110
      IF(IFMT.EQ.3111) GO TO 3111
3000  WRITE(6,1) NTABLE,CSQP,CSQB,CSQG,(((IX(I,J,K),J=1,J1),I=1,I1),K=1,
     5K1),IADF
      WRITE(8,9)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSQP,CS
     5QB,CSQG,(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
      RETURN
3001  WRITE(8,2)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSQP,C
     6SQB,(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
      IF(NREJ.EQ.0.OR.NREJ.EQ.2) RETURN
      WRITE(6,11)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,NTABLE,CSQP,CSQB,
     6(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1),IADF
      RETURN
3010  WRITE(8,3)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSQP,C
     7SQG,(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
      IF(NREJ.EQ.0.OR.NREJ.EQ.2) RETURN
      WRITE(6,12)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,NTABLE,CSQP,CSQG,
     7(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1),IADF
      RETURN
3100  WRITE(8,4)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSQB,C
     8SQG,(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
      IF(NREJ.EQ.0.OR.NREJ.EQ.2) RETURN
      WRITE(6,13)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,NTABLE,CSQB,CSQG,
     8(((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1),IADF
      RETURN
3011  WRITE(8,5)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSQP,(
     9((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
```

```
      RETURN
3101  WRITE(3,6)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSUB,(
     9((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
      RETURN
3110  WRITE(3,7)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,CSUG,(
     9((IX(I,J,K),J=1,J1),I=1,I1),K=1,K1)
      RETURN
3111  WRITE(3,8)IRN,IDIM,NPROPS,LR,IFRMP,IFRMB,IFRMG,IADF,NTABLE,(((IX(
     9I,J,K),J=1,J1),I=1,I1),K=1,K1)
      RETURN
1     FORMAT(2X,I4,1X,3(F7.2,1X),27I3,5X,I3)
2     FORMAT(3I2,5I3,I4,2F7.2,7X,27I2)
3     FORMAT(3I2,5I3,I4,F7.2,7X,F7.2,27I2)
4     FORMAT(3I2,5I3,I4,7X,2F7.2,27I2)
5     FORMAT(3I2,5I3,I4,F7.2,14X,27I2)
6     FORMAT(3I2,5I3,I4,7X,F7.2,7X,27I2)
7     FORMAT(3I2,5I3,I4,14X,F7.2,27I2)
8     FORMAT(3I2,5I3,I4,21X,27I2)
9     FORMAT(3I2,5I3,I4,3F7.2,27I2)
11    FORMAT(2X,3I2,4I3,I4,2F7.2,7X,27I2,5X,I3)
12    FORMAT(2X,3I2,4I3,I4,F7.2,7X,F7.2,27I2,I3)
13    FORMAT(2X,3I2,4I3,I4,7X,2F7.2,27I2,5X,I3)
      END
      SUBROUTINE DECOMP(NN,A,UL,IFLAG)
      DIMENSION A(12,12),UL(12,12),SCALES(12),IPS(12)
      DOUBLE PRECISION A,UL,ROWNRM,SCALES,BIG,SIZE,PIVOT,EM
      COMMON IPS
      N=NN
      DO 5 I=1,N
      IPS(I)=I
      ROWNRM=0.0
      DO 2 J=1,N
      UL(I,J)=A(I,J)
      IF(ROWNRM-DABS(UL(I,J)))1,2,2
1     ROWNRM=DABS(UL(I,J))
2     CONTINUE
      IF(ROWNRM)3,4,3
3     SCALES(I)=1./ROWNRM
      CALL OVERFL(IFLO#)
      IF(IFLO# .NE. 2) GO TO 4
      GO TO 5
4     IFLAG=1
      RETURN
5     CONTINUE
      NM1=N-1
      DO 17 K=1,NM1
      BIG=0.0
      DO 11 I=K,N
      IP=IPS(I)
      SIZE=DABS(UL(IP,K))*SCALES(IP)
      IF(SIZE-BIG)11,11,10
10    BIG=SIZE
      IDXPIV=I
11    CONTINUE
      IF(BIG) 13,12,13
12    IFLAG=1
      RETURN
13    IF(IDXPIV-K)14,15,14
14    J=IPS(K)
      IPS(K)=IPS(IDXPIV)
      IPS(IDXPIV)=J
15    KP=IPS(K)
```

```
      PIVOT=UL(KP,K)
      KP1=K+1
      DO 16 I=KP1,N
      IP=IPS(I)
      EM=-UL(IP,K)/PIVOT
      CALL OVCHK(IDIV)
      CALL OVERFL(IFLOW)
      IF(IDIV .EQ. 2 .AND. IFLOW .EQ. 2) GO TO 20
      IFLAG=1
      RETURN
20    UL(IP,K)=-EM
      DO 16 J=KP1,N
      UL(IP,J)=UL(IP,J)+EM*UL(KP,J)
16    CONTINUE
17    CONTINUE
      KP=IPS(N)
      IF(UL(KP,N))19,18,19
18    IFLAG=1
19    RETURN
      END
      SUBROUTINE SOLVE(NN,UL,B,X,IFLAG)
      DIMENSION UL(12,12),B(12),X(12),IPS(12)
      COMMON IPS
      DOUBLE PRECISION UL,B,X,SUM
      N=NN
      IFLOW=0
      NP1=N+1
      IP=IPS(1)
      X(1)=B(IP)
      DO 2 I=2,N
      IP=IPS(I)
      IM1=I-1
      SUM=0.0
      DO 1 J=1,IM1
1     SUM=SUM+UL(IP,J)*X(J)
2     X(I)=B(IP)-SUM
      IP=IPS(N)
      X(N)=X(N)/UL(IP,N)
      CALL OVERFL(IFLOW)
      CALL OVCHK(IDIV)
      IF(IFLOW.EQ.2 .AND. IDIV .EQ. 2) GO TO 10
      IFLAG=1
      RETURN
10    DO 4 IBACK=2,N
      I=NP1-IBACK
      IP=IPS(I)
      IP1=I+1
      SUM=0.
      DO 3 J=IP1,N
3     SUM=SUM+UL(IP,J)*X(J)
      X(I)=(X(I)-SUM)/UL(IP,I)
      CALL OVCHK(IDIV)
      CALL OVERFL(IFLOW)
      IF (IFLOW.EQ.2 .AND. IDIV .EQ. 2) GO TO 4
      IFLAG=1
      RETURN
4     CONTINUE
      RETURN
      END
      SUBROUTINE IMPROV(NN,A,UL,B,X,DIGITS,IFLAG)
      DIMENSION A(12,12),UL(12,12),B(12),X(12),R(12),DX(12)
      DOUBLE PRECISION A,UL,B,X,R,DX,DIGITS,SUM,EPS,XNORM,DXNORM,T
```

```
      N=NN
      EPS=10.E-14
      ITMAX=50
      XNORM=0.0
      DO 1 I=1,N
1     XNORM=DMAX1(XNORM,DABS(X(I)))
      IF(XNORM)3,2,3
2     DIGITS=-DLOG10(EPS)
      GO TO 10
3     DO 9 ITER=1,ITMAX
      DO 5 I=1,N
      SUM=0.0
      DO 4 J=1,N
4     SUM=SUM+A(I,J)*X(J)
      SUM=B(I)-SUM
5     R(I)=SUM
      CALL SOLVE(N,UL,R,DX,IFLAG)
      IF(IFLAG.EQ.1) GO TO 10
      DXNORM=0.0
      DO 6 I=1,N
      T=X(I)
      X(I)=X(I)+DX(I)
      DXNORM=DMAX1(DXNORM,DABS(X(I)-T))
6     CONTINUE
      IF(ITER-1)8,7,8
7     DIGITS=-DLOG10(DMAX1(DXNORM/XNORM,EPS))
8     IF(DXNORM-EPS*XNORM)10,10,9
9     CONTINUE
      IFLAG=1
10    RETURN
      END
      SUBROUTINE BARTCH(I1,J1,K1,X,NCHB)
      DIMENSION X(3,3,3),XX(3,3,3)
      DOUBLE PRECISION X,XX
      NCHB=NCHB+1
      GO TO (1,2,2,1,2,2,1,2),NCHB
1     DO 20 K=1,K1
      DO 20 J=1,J1
      IF(J.EQ.J1) GO TO 5
      DO 10 I=1,I1
10    XX(I,J,K)=X(I,J+1,K)
      GO TO 20
5     DO 15 I=1,I1
15    XX(I,J,K)=X(I,1,K)
20    CONTINUE
      GO TO 25
2     DO 30 K=1,K1
      DO 30 I=1,I1
      IF(I.EQ.I1) GO TO 35
      DO 40 J=1,J1
40    XX(I,J,K)=X(I+1,J,K)
      GO TO 30
35    DO 45 J=1,J1
45    XX(I,J,K)=X(1,J,K)
30    CONTINUE
25    DO 50 I=1,I1
      DO 50 J=1,J1
      DO 50 K=1,K1
50    X(I,J,K)=XX(I,J,K)
      RETURN
      END
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APPROVAL SHEET

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The final copies have been examined by the director of the dissertation and the signature which appears below verifies the fact that any necessary changes have been incorporated and that the dissertation is now given final approval by the Committee with reference to content and form.

The dissertation is therefore accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April 21, 1971
Date

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